INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Camera Self-Calibration from Video Sequences: the Kruppa Equations Revisited

Cyril Zeller, Olivier Faugeras

INRIA Sophia Antipolis, BP 93, 06902 Sophia Antipolis Cedex, France

N° 2793

Février 1996

PROGRAMME 4

Rapport de recherche

ISSN 0249-6399

Camera Self-Calibration from Video Sequences: the Kruppa Equations Revisited

Cyril Zeller, Olivier Faugeras
INRIA Sophia Antipolis, BP 93, 06902 Sophia Antipolis Cedex, France

Programme 4 — Robotique, image et vision
Projet Robotvis

Rapport de recherche n° 2793 — Février 1996 — 65 pages

Abstract: The use of some new techniques such as self-calibration and calibration from partial a priori information about the observed scene (angles, distances), together with the use of some classical techniques of video sequence processing, such as features extraction and token tracking, makes the computation of three-dimensional measurements of the observed scene from video sequences possible. Depending upon the kind of calibration that has been obtained for the camera, these measurements are projective, affine or metric measurements. They can be obtained directly from the images without any reconstruction process.

The self-calibration technique described in this article is the generalization to a large number of images of the algorithm developped by Luong and Faugeras[8] based on the Kruppa equations. A theoretical study of these equations as well as experimental results on real video sequences are presented.

Key-words: self-calibration, measurements from video sequences.

(Résumé : tsvp)

Unité de recherche INRIA Sophia-Antipolis
2004 route des Lucioles, BP 93, 06902 SOPHIA-ANTIPOLIS Cedex (France)
Téléphone : (33) 93 65 77 77 – Télécopie : (33) 93 65 77 65

Autocalibration à partir d'une séquence vidéo: les équations de Kruppa revisitées.

Résumé : L'utilisation de nouvelles techniques, telles que l'autocalibration et la calibration à partir d'informations connues a priori sur la scène observée (angles, distances), conjointement à celle de techniques classiques de traitement d'image, telles que l'extraction de primitives caractéristiques et leur suivi, rend possible le calcul de mesures tridimensionnelles de la scène observée. Suivant le type de la calibration qui a été obtenue pour le système, ces mesures sont de type projectif, affine ou métrique. Elles peuvent être calculées directement dans les images sans passer par une reconstruction de la scène.

La technique d'autocalibration décrite dans cet article est la généralisation à un grand nombre d'images de l'algorithme développé par Luong et Faugeras[8] basé sur les équations de Kruppa. Une étude théorique de ces équations ainsi que des résultats expérimentaux sur des séquences vidéo réelles sont présentées.

Mots-clé : autocalibration, mesures à partir de séquence vidéos.

Contents

1 Introduction **3**

2 The model **3**
2.1 The camera 3
2.2 The scene 4

3 Two views **4**
3.1 The disparity between two views 5
3.1.1 The general case 5
3.1.2 The case of coplanar points 6
3.1.3 The case of points at infinity 7
3.2 Measurements from two views 8
3.2.1 Projective measurements 8
3.2.2 Affine measurements 8
3.2.3 Metric measurements 9

4 Computing F and $\mathbf{H}_\infty$ from point correspondences **10**

5 Computing A from $\mathbf{H}_\infty$ or F **11**
5.1 Notations 11
5.2 Computing **A** from $\mathbf{H}_\infty$ 11
5.3 Computing **A** from **F** 12
5.3.1 The Kruppa equations 13
5.3.2 A study of the Kruppa equations 15
5.3.3 Solving the Kruppa equations 15
5.4 Using geometric information 17

6 Computing the projection matrices from A and F **18**

7 Results **20**

8 Conclusion **37**

A Study of $\mathbf{S}_\lambda$ **39**

B Study of the self-calibration equations **43**
B.1 Notations 43
B.2 The solutions corresponding to one displacement 44
B.3 The solutions corresponding to two displacements 48

C Study of ranks **51**

D Harmonic vectors **61**

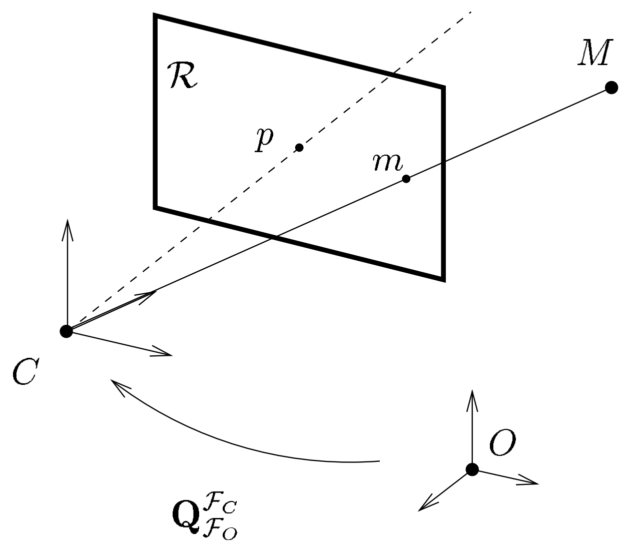


Figure 1: The pinhole model.

1 Introduction

Section 2 describes the model used for the camera and the scene observed and section 3 analyses the relationship between two views. Some calibration parameters naturally appear in section 3.1: the fundamental matrix $\mathbf{F}$, the homography of the plane at infinity $\mathbf{H}_\infty$ and the intrinsic parameters matrices $\mathbf{A}$ and $\mathbf{A}'$. The knowledge of these quantities allows to compute projective, affine or metric measurements as described in section 3.2.

Section 4 briefly explains how to compute $\mathbf{F}$ and $\mathbf{H}_\infty$ from point correspondences. Section 5 explains how, in the case where $\mathbf{A}' = \mathbf{A}$, the camera self-calibrates from a video sequence by computing $\mathbf{A}$ from $\mathbf{F}$ or $\mathbf{H}_\infty$. Some partial a priori information about the scene observed may easily be introduced in the process of self-calibration.

2 The model

2.1 The camera

The camera model used is the classical *pinhole model*. If the object space is considered to be the 3-dimensional Euclidean space $\mathcal{R}^3$ embedded in the usual way in the 3-dimensional projective space $\mathcal{P}^3$ and the image space the 2-dimensional Euclidean space $\mathcal{R}^2$ embedded in the usual way in the 2-dimensional projective space $\mathcal{P}^2$, the camera is then described as a *linear projective application* from $\mathcal{P}^3$ to $\mathcal{P}^2$[6]. We can write the projection matrix in any object frame $\mathcal{F}_O$ of $\mathcal{P}^3$:

$$\underbrace{\begin{bmatrix} \alpha_u & \gamma & u_0 \\ 0 & \alpha_v & v_0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{P}_0} \mathbf{Q}_{\mathcal{F}_O}^{\mathcal{F}_C} \tag{1}$$

where $\mathbf{A}$ is the matrix of the *intrinsic parameters*, C the optical center (see figure 1) and $\mathbf{Q}_{\mathcal{F}_i}^{\mathcal{F}_j}$ is the notation for the matrix of change of frame $\mathcal{F}_i$ to frame $\mathcal{F}_j$, such that $\mathbf{M}_{/\mathcal{F}_j} = \mathbf{Q}_{\mathcal{F}_i}^{\mathcal{F}_j} \mathbf{M}_{/\mathcal{F}_i}$.

In particular, the projection equation, relating a point not in the focal plane $\mathbf{M}_{/\mathcal{F}_C}^T = [X_C, Y_C, Z_C, T_C]^T$, expressed in the normalized camera frame to its projection $\mathbf{m}^T = [x, y, 1]^T$ is

$$Z_C \mathbf{m} = \mathbf{A}\mathbf{P}_0 \mathbf{M}_{/\mathcal{F}_C} \tag{2}$$

2.2 The scene

Even though our formalism also applies to dynamic objects, we concentrate in this paper on scenes composed of static objects. Moreover, when we study the disparity between two views, this restriction does not appear as a restriction any more if the two views have been taken simultaneously by a stereoscopic system.

3 Two views

We study the relationship between two views of a scene. These views are supposed to come from either two cameras or one camera in motion. The optical centers corresponding to the views are denoted by C for the first and C' for the second, the intrinsic parameters by $\mathbf{A}$ and $\mathbf{A}'$ respectively, the normalized camera frames respectively by $\mathcal{F}_C$ and $\mathcal{F}_{C'}$. The matrix of change of frame $\mathcal{F}_C$ to frame $\mathcal{F}_{C'}$ is a matrix of displacement defined by a rotation matrix $\mathbf{R}$ and a translation vector $\mathbf{t}$:

$$\mathbf{Q}_{\mathcal{F}_C}^{\mathcal{F}_{C'}} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}_3^T & 1 \end{bmatrix} \tag{3}$$

More precisely, given a point M of an object o, we are interested in establishing the disparity equation of M for the two views, that is the equation relating the projection m' of M in the second view to the projection m of M in the first view.

3.1 The disparity between two views

3.1.1 The general case

Assuming that M is not in the focal planes corresponding to the first and second views, we have, from equations (2) and (3):

$$Z'_{C'}\mathbf{m}' = \mathbf{A}'\mathbf{P}_0\mathbf{M}'_{/\mathcal{F}_{C'}} = \mathbf{A}' \begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix} \mathbf{M}_{/\mathcal{F}_C} = Z_C\mathbf{A}'\mathbf{R}\mathbf{A}^{-1}\mathbf{m} + T_C\mathbf{A}'\mathbf{t}$$

We thus have obtained the general *disparity equation* relating m' to m:

$$Z'_{C'}\mathbf{m}' = Z_C\mathbf{H}_\infty\mathbf{m} + T_C\mathbf{e}' \tag{4}$$

where

$$\mathbf{H}_\infty = \mathbf{A}'\mathbf{R}\mathbf{A}^{-1} \tag{5}$$

$$\mathbf{e}' = \mathbf{A}'\mathbf{t} \tag{6}$$

$\mathbf{H}_\infty$ is the *homography of the plane at infinity*, as detailed below in section 3.1.3. $\mathbf{e}'$ is a vector representing the *epipole* in the image frame of the second view, that is, the projection of C in the second view. Indeed, this projection is

$$\mathbf{A}'\mathbf{P}_0\mathbf{C}_{/\mathcal{F}_{C'}} = \mathbf{A}' \begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix} \mathbf{C}_{/\mathcal{F}_C} = \mathbf{A}'\mathbf{t}$$

Similarly,

$$\mathbf{e} = -\mathbf{A}\mathbf{R}^T\mathbf{t} \tag{7}$$

is a vector representing the epipole in the image frame of the first view.

Equation (4) means that m' lies on the line going through e' and the point represented by $\mathbf{H}_\infty\mathbf{m}$, which is the *epipolar line* of m. It is given by the vector

$$\mathbf{F}\mathbf{m} \tag{8}$$

where

$$\mathbf{F} = [\mathbf{e}']_\times\mathbf{H}_\infty \tag{9}$$

or equivalently [1],

$$\mathbf{F} = \det(\mathbf{A}')\mathbf{A}'^{-1T}\mathbf{E}\mathbf{A}^{-1} \tag{10}$$

[1] using the algebraic equation $[\mathbf{M}\mathbf{u}]_\times = \det(\mathbf{M})\mathbf{M}^{-1T}[\mathbf{u}]_\times\mathbf{M}^{-1}$, valid if $\det(\mathbf{M}) \neq 0$

with

$$\mathbf{E} = [\mathbf{t}]_{\times}\mathbf{R} \tag{11}$$

$\mathbf{F}$ is the *fundamental matrix* which describes the correspondence between an image point in the first view and its epipolar line in the second (see [8]). $\mathbf{E}$ is the *essential matrix*[6].

Given two projection matrices $\begin{bmatrix} \mathbf{P} & \mathbf{p} \end{bmatrix}$ and $\begin{bmatrix} \mathbf{P}' & \mathbf{p}' \end{bmatrix}$ expressed in the same object frame $\mathcal{F}_O$, we can easily write the disparity equation: If the projection equations are

$$\lambda_{\mathbf{m}}\mathbf{m} = \mathbf{P}\mathbf{P}_0\mathbf{M}_{/\mathcal{F}_O} + \mathbf{p} \quad \text{and} \quad \lambda'_{\mathbf{m}'}\mathbf{m}' = \mathbf{P}'\mathbf{P}_0\mathbf{M}_{/\mathcal{F}_O} + \mathbf{p}'$$

the disparity equation is written

$$\lambda'_{\mathbf{m}'}\mathbf{m}' = \lambda_{\mathbf{m}}\mathbf{P}'\mathbf{P}^{-1}\mathbf{m} + \mathbf{p}' - \mathbf{P}'\mathbf{P}^{-1}\mathbf{p} \tag{12}$$

This implies that $\mathbf{e}'$ is proportional to $\mathbf{p}' - \mathbf{P}'\mathbf{P}^{-1}\mathbf{p}$, and $\mathbf{F}$ to $[\mathbf{p}' - \mathbf{P}'\mathbf{P}^{-1}\mathbf{p}]_{\times}\mathbf{P}'\mathbf{P}^{-1}$.

3.1.2 The case of coplanar points

In the case of coplanar points, the equation of the plane in $\mathcal{F}_C$, relating Z_C and T_C, allows us to unify their different disparity equations in one disparity equation valid for all of them.

The plane being given in $\mathcal{F}_C$ by the vector $\mathbf{\Pi}^T = \begin{bmatrix} \mathbf{n}^T & -d \end{bmatrix}$, where $\mathbf{n}$ is its unitary normal in $\mathcal{F}_C$ and d, the distance from the plane to C, its equation is $\mathbf{\Pi}^T\mathbf{M}_{/\mathcal{F}_C} = 0$, which can be written, using equation (2),

$$\begin{aligned} 0 &= \mathbf{n}^T\mathbf{P}_0\mathbf{M}_{/\mathcal{F}_C} - T_C d && (13) \\ &= Z_C\mathbf{n}^T\mathbf{A}^{-1}\mathbf{m} - T_C d && (14) \end{aligned}$$

If we first assume that $d \neq 0$, that is the plane does not go through C, we then obtain the new form of the disparity equation[2]:

$$Z'_{C'}\mathbf{m}' = Z_C\mathbf{H}\mathbf{m} \tag{15}$$

where

$$\mathbf{H} = \mathbf{H}_{\infty} + \mathbf{e}'\frac{\mathbf{n}^T}{d}\mathbf{A}^{-1} \tag{16}$$

[2] using the algebraic equation $\mathbf{u}\mathbf{v}^T\mathbf{M}\mathbf{w} = (\mathbf{v}^T\mathbf{M}\mathbf{w})\mathbf{u}$

This equation establishes the linear projective application, given by $\mathbf{H}$, the *H-matrix* of the plane, relating the projections of the points of the plane in the first view to their projections in the second. It is at the basis of the idea which consists in segmenting the scene in planar structures given by their respective H-matrices and, using this segmentation, to compute motion and structure (see [4] or [18]).

If the plane does not go either through C', its H-matrix is a homography ($\det(\mathbf{H}) \neq 0$) since its inverse is given by

$$\mathbf{H}^{-1} = \mathbf{H}' = \mathbf{H}_\infty^{-1} + \mathbf{e}\frac{\mathbf{n}'^T}{d'}\mathbf{A}'^{-1} \tag{17}$$

where $\mathbf{n}'$ is its unitary normal in $\mathcal{F}_{C'}$ and d', the distance from the plane to C'. If the plane goes through only one of the two points C or C', its H-matrix is still defined by the one of the two equations (16) or (17) which remains valid, but is no longer a homography; equation (14) shows that the plane then projects in one of the two views in a line representing by the vector

$$\mathbf{A}^{-1T}\mathbf{n} \quad \text{or} \quad \mathbf{A}'^{-1T}\mathbf{n}' \tag{18}$$

If the plane is an epipolar plane, that is goes through both C and C', its H-matrix is undefined.

Equations (16) and (9) show that we have

$$\mathbf{F} = [\mathbf{e}']_\times \mathbf{H} \tag{19}$$

Finally, equation (7) shows that e' and e always satisfy equation (15), as expected, since e' and e are the projections of the intersection of the line $< CC' >$ with the plane.

3.1.3 The case of points at infinity

For the points of the plane at infinity, represented by $[0,0,0,1]^T$, of equation $T_C = 0$, the disparity equation becomes

$$Z'_{C'}\mathbf{m}' = Z_C \mathbf{H}_\infty \mathbf{m} \tag{20}$$

Thus, $\mathbf{H}_\infty$ is indeed the H-matrix of the plane at infinity. Equation (20) is also the limit of equation (15), when $d \to \infty$, which is compatible with the fact that the points at infinity correspond to the remote points of the scene.

3.2 Measurements from two views

It has been shown that the knowledge of $\mathbf{F}$ allows to compute a projective reconstruction of the scene [5, 12]. Likewise, the knowledge of $\mathbf{F}$ and $\mathbf{H}_\infty$ leads to an affine reconstruction and the knowledge of $\mathbf{F}$, $\mathbf{H}_\infty$, $\mathbf{A}$ and $\mathbf{A}'$ leads to an Euclidean reconstruction[19, 20]. In each of these cases, some measurements can be directly computed in the images.

3.2.1 Projective measurements

The camera is modeled as a projective linear transformation so some projective measurements are computable without any knowledge. It is the case for the cross-ratio of four aligned points which is simply equal to the cross-ratio of their images.

If $\mathbf{F}$ is known, the H-matrix of a plane defined by any three points is computable, as explained in section 4. It is then possible to compute the intersection between planes and lines.

Given the H-matrix $\mathbf{H}$ of a plane P and the correspondences (m, m') and (n, n') of two points M and N, the image i' of the intersection of the line $< MN >$ with P is

$$\mathbf{i}' = (\mathbf{m}' \times \mathbf{n}') \times (\mathbf{Hm} \times \mathbf{Hn})$$

since i' belongs both to $< m'n' >$ and the image of $< mn >$ by $\mathbf{H}$[16].

Similarly, given two planes P and Q by their H-matrices $\mathbf{H}_P$ and $\mathbf{H}_Q$, the correspondences of the intersection L of P with Q is obtained by computing the correspondences of two points of L. These points are, for example, the intersections of two lines L_1 and L_2 of P with Q. The correspondences of such lines are obtained by choosing two lines in the first image representing by the vectors $\mathbf{l}_1$ and $\mathbf{l}_2$, their corresponding lines in the second image being given by $\mathbf{H}_P^{-1T}\mathbf{l}_1$ and $\mathbf{H}_P^{-1T}\mathbf{l}_2$.

3.2.2 Affine measurements

If $\mathbf{F}$ and $\mathbf{H}_\infty$ are known, we can compute the intersection of any line with the plane at infinity, as explained in section 3.2.1. This allows us to tell whether two lines are parallel or not: they are parallel if and only if they intersect at the same point at infinity. This also allows us to compute the ratio $\frac{M_1M_2}{M_1M_3}$ for any three points M_1, M_2 and M_3 belonging to a line L, as the cross-ratio of M_1, M_2, M_3 and the intersection of L with the plane at infinity.

3.2.3 Metric measurements

If $\mathbf{F}$, $\mathbf{H}_\infty$, $\mathbf{A}$ and $\mathbf{A}'$ are known, it is possible to compute the angle between two lines or the ratio of two lengths[7, 2].

Angle between lines. The angle between two lines L_1 and L_2 is given by the Laguerre formula

$$(L_1, L_2) = \frac{1}{2i}\log(\{V_1, V_2, I_{12}, J_{12}\})$$

where V_1 and V_2 are the intersections with the plane at infinity of L_1 and L_2 respectively, and I_{12} and J_{12}, the intersections of $< V_1V_2 >$ with the *absolute conic* Ω of equation

$$X^2 + Y^2 + Z^2 = 0 \quad , \quad T = 0$$

in any orthonormal frame of $\mathcal{P}^3$. $\{V_1, V_2, I_{12}, J_{12}\}$ denotes the cross-ratio of these four points so that we also have

$$(L_1, L_2) = \frac{1}{2i}\log(\{v_1, v_2, i_{12}, j_{12}\}) \tag{21}$$

v_1 and v_2 are computed using $\mathbf{H}_\infty$, as explained in section 3.2.1. i_{12} and j_{12} are computed as the intersections of $< v_1v_2 >$ with the image ω of Ω. According to equation (2), the equation of ω is

$$S(\mathbf{m}, \mathbf{m}) = 0 \tag{22}$$

where

$$S(\mathbf{p}, \mathbf{q}) = \mathbf{p}^T\mathbf{A}^{-1T}\mathbf{A}^{-1}\mathbf{q}$$

A general point of $< v_1v_2 >$ is $\mathbf{v}_1 + \theta\mathbf{v}_2$, considering that v_2 corresponds to $\theta = \infty$. According to equation (22), i_{12} and j_{12} correspond to the two complex conjugate roots θ_0 and $\bar{\theta}_0$ of

$$S(\mathbf{v}_1, \mathbf{v}_1) + 2\theta S(\mathbf{v}_1, \mathbf{v}_2) + S(\mathbf{v}_2, \mathbf{v}_2)\theta^2 = 0$$

so that

$$\theta_0 = \frac{-S(\mathbf{v}_1, \mathbf{v}_2) \pm i\sqrt{S(\mathbf{v}_1, \mathbf{v}_1)S(\mathbf{v}_2, \mathbf{v}_2) - S(\mathbf{v}_1, \mathbf{v}_2)^2}}{S(\mathbf{v}_2, \mathbf{v}_2)} \tag{23}$$

Now, we have

$$\{v_1, v_2, i_{12}, j_{12}\} = \frac{0 - \theta_0}{0 - \bar{\theta}_0} : \frac{\infty - \theta_0}{\infty - \bar{\theta}_0} = \frac{\theta_0}{\bar{\theta}_0}$$

We deduce from this and equations (21) and (23) that $\cos(L_1, L_2)$ is given by

$$c(\mathbf{v}_1, \mathbf{v}_2) = -\frac{S(\mathbf{v}_1, \mathbf{v}_2)}{\sqrt{S(\mathbf{v}_1, \mathbf{v}_1)S(\mathbf{v}_2, \mathbf{v}_2)}} \tag{24}$$

Ratio of lengths. The ratio $\frac{AB}{CD}$, where A, B, C and D are four points, decomposed as

$$\frac{AB}{CD} = \frac{AB}{BC}\frac{BC}{CD} = \frac{\sin(\vec{CA}, \vec{CB})}{\sin(\vec{AB}, \vec{AC})}\frac{\sin(\vec{DB}, \vec{DC})}{\sin(\vec{BC}, \vec{BD})}$$

So, once computed the images v_{ab}, v_{ac}, v_{bc}, v_{bd} and v_{cd} of the intersections with the plane at infinity of $< AB >$, $< AC >$, $< BC >$, $< BD >$ and $< CD >$, respectively, as explained in section 3.2.1, $\frac{AB}{CD}$ is given by

$$r(\mathbf{v}_{ab}, \mathbf{v}_{ac}, \mathbf{v}_{bc}, \mathbf{v}_{bd}, \mathbf{v}_{cd}) =$$
$$\sqrt{\frac{S(\mathbf{v}_{ab}, \mathbf{v}_{ab})}{S(\mathbf{v}_{cd}, \mathbf{v}_{cd})} \cdot \frac{S(\mathbf{v}_{ac}, \mathbf{v}_{ac})S(\mathbf{v}_{bc}, \mathbf{v}_{bc}) - S^2(\mathbf{v}_{ac}, \mathbf{v}_{bc})}{S(\mathbf{v}_{ab}, \mathbf{v}_{ab})S(\mathbf{v}_{ac}, \mathbf{v}_{ac}) - S^2(\mathbf{v}_{ab}, \mathbf{v}_{ac})} \cdot \frac{S(\mathbf{v}_{bd}, \mathbf{v}_{bd})S(\mathbf{v}_{cd}, \mathbf{v}_{cd}) - S^2(\mathbf{v}_{bd}, \mathbf{v}_{cd})}{S(\mathbf{v}_{bc}, \mathbf{v}_{bc})S(\mathbf{v}_{bd}, \mathbf{v}_{bd}) - S^2(\mathbf{v}_{bc}, \mathbf{v}_{bd})}} \tag{25}$$

which results from equation (24).

4 Computing F and $\mathbf{H}_\infty$ from point correspondences

F and the H-matrix **H** of any plane P, and in particular $\mathbf{H}_\infty$, are computed from image point correspondences between the two views. These correspondences are obtained either by tracking if the views belong to a sequence or by stereo matching[19, 20, 21, 3].

The disparity equation gives then the equations relating **F** or **H** to the image point correspondences:

$$\mathbf{m}'^T\mathbf{F}\mathbf{m} = 0 \tag{26}$$
$$\mathbf{m}' \times \mathbf{H}\mathbf{m} = 0 \tag{27}$$

We notice that these equations allow to compute **F** and **H** only up to a non-zero scalar factor. So, eight points for **F** and four points for **H** in general configuration are enough to compute them linearly. The epipoles are then computed as the eigenvectors of **F** and $\mathbf{F}^T$ corresponding to the 0 eigenvalue. According to section 3.1.2, they satisfy equation (27) and so three other points are enough to compute **H**.

In practice, much more points are available so that $\mathbf{F}$ and $\mathbf{H}$ are computed as solutions of least squares problems [14, 21]. Their covariance matrices $\mathbf{\Lambda_F}$ and $\mathbf{\Lambda_H}$ are also computed in order to quantify their uncertainty[3].

Of course, since equation (27) is satisfied only by some image point correspondences corresponding to points belonging to P, some of them must be known to make the computation of $\mathbf{H}$ possible. In particular, to compute $\mathbf{H}_\infty$, we need to know images of points at infinity, such as the intersections of parallel lines[17, 19, 20]. On the contrary, equation (26) is satisfied by any image point correspondences, which makes the computation of $\mathbf{F}$ more immediate, since no additional knowledge is necessary.

5 Computing A from $\mathbf{H}_\infty$ or F

While $\mathbf{F}$ is more easily computable than $\mathbf{H}_\infty$, it is in return more difficult to compute $\mathbf{A}$ from $\mathbf{F}$ than from $\mathbf{H}_\infty$, as explained in section 5.2 and 5.3.

5.1 Notations

The set of all the symmetric 3×3-matrices is a 6-dimensional vector space and $\mathbf{X}$ equally refers to the matrix or its corresponding 6-dimensional vector.

For a set of linearly independent vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_N\}$, $\mathcal{V}(\mathbf{v}_1, \dots, \mathbf{v}_N)$ denotes the vector space of basis $(\mathbf{v}_1, \dots, \mathbf{v}_N)$.

We denote the rotation axis of a rotation matrix $\mathbf{R}$ by any vector $\mathbf{r}$, such that $\mathbf{Rr} = \mathbf{r}$.

5.2 Computing A from $\mathbf{H}_\infty$

Writing that $\mathbf{R}$ is an orthogonal matrix and using equation (5), we have

$$\mathbf{RR}^T = \mathbf{I}_3 \Longleftrightarrow \mathbf{A}'^{-1}\mathbf{H}_\infty \mathbf{AA}^T \mathbf{H}_\infty^T \mathbf{A}'^{-1T} = \mathbf{I}_3$$

which leads to

$$\mathbf{H}_\infty \mathbf{K} \mathbf{H}_\infty^T = \mathbf{K}' \tag{28}$$

where

$$\mathbf{K} = \mathbf{AA}^T \quad \text{and} \quad \mathbf{K}' = \mathbf{A}'\mathbf{A}'^T$$

$\mathbf{K}$ is the matrix of the *Kruppa coefficients* related to $\mathbf{A}$. If we consider that α_u and α_v (see equation (1)) are different from zero, which is always the case in practice, then

$\mathbf{K}$ is a positive definite symmetric matrix: It is thus in one-to-one correspondence with $\mathbf{A}$ since $\mathbf{A}$ is uniquely obtained by constructing the Cholesky decomposition of $\mathbf{K}$.

Equation (28) shows how we can compute $\mathbf{A}'$ from $\mathbf{A}$ and $\mathbf{H}_\infty$.

In the case where $\mathbf{A}' = \mathbf{A}$, we can use equation (28) to compute $\mathbf{A}$ from $\mathbf{H}_\infty$. Since $\det(\mathbf{H}_\infty) = \det(\mathbf{A}) \det(\mathbf{R} \det(\mathbf{A}^{-1})$ is then equal to 1, $\mathbf{H}_\infty$ is exactly known without a scalar factor indetermination and the problem is linear. More precisely, proposition B.2 shows that a non-zero symmetric matrix $\mathbf{X}$ is a solution of equation

$$\mathbf{H}_\infty \mathbf{X} \mathbf{H}_\infty^T = \mathbf{X} \tag{29}$$

if and only if

$$\mathbf{X} \in \mathcal{V}(\mathbf{A}\mathbf{A}^T, \mathbf{A}\mathbf{r}\mathbf{r}^T\mathbf{A}^T)$$

This has already been shown [18, 13]. Consequently, two equations of the same type as equation (29), obtained, for example, from three views with same intrinsic parameters $\mathbf{A}$, are necessary and sufficient to compute $\mathbf{A}$, as soon as their corresponding rotation axis are not parallel (see proposition B.7).

In practice, more than three views are available, so that $\mathbf{K}$ is computed as the solution of a linear least squares problem [18, 1, 15].

5.3 Computing A from F

By multiplying equation (28) on the lefthand side by $[\mathbf{e}']_\times$ and on the righthand side by $[\mathbf{e}']_\times^T$ and using equation (9), we obtain

$$\mathbf{F}\mathbf{K}\mathbf{F}^T = [\mathbf{e}']_\times \mathbf{K}' [\mathbf{e}']_\times^T \tag{30}$$

But equation (30) is not usable as such in practice. Since $\mathbf{F}$ and $\mathbf{e}'$ are computed only up to non-zero scalar factors λ and μ (see section 4)

$$\tilde{\mathbf{F}} = \lambda \mathbf{F} \quad \text{and} \quad \tilde{\mathbf{e}}' = \mu \mathbf{e}' \tag{31}$$

the equation actually available to us is

$$\tilde{\mathbf{F}}\mathbf{K}\tilde{\mathbf{F}}^T = k^2 [\tilde{\mathbf{e}}']_\times \mathbf{K}' [\tilde{\mathbf{e}}']_\times^T \quad \text{with} \quad k = \frac{\lambda}{\mu} \tag{32}$$

Equation (32) shows that we cannot compute $\mathbf{A}'$ from $\mathbf{A}$ and $\mathbf{F}$ since, according to proposition B.3, the following homogeneous linear system in the symmetric matrix $\mathbf{X}$

$$[\mathbf{e}']_\times \mathbf{X} [\mathbf{e}']_\times^T = \mathbf{0}_{3\times 3}$$

is of rank 3.

In the case where $\mathbf{A}' = \mathbf{A}$, we can use equation (32) to compute $\mathbf{A}$ from $\mathbf{F}$. Since $\mathbf{e}'$ is computed as the eigenvector of $\mathbf{F}^T$ corresponding to the 0 eigenvalue, k is also unknown and the problem is not linear.

Section 5.3.1 puts the problem in terms of the well-known Kruppa equations [8] which shows that the problem is quadratic indeed. Section 5.3.2, together with appendices A, B, C and D, gives the formal solutions of the Kruppa equations in function of $\mathbf{A}$, $\mathbf{R}$ and $\mathbf{t}$. This allows us to show off the particular spatial configurations of the cameras that lead to degenerate cases for the Kruppa equations. Lastly, section 5.3.3 describes how we solve the Kruppa equations in practice.

5.3.1 The Kruppa equations

The problem of recovering $\mathbf{A}$ from $\mathbf{F}$ using equation (32) amounts to find the symmetric matrices $\mathbf{X}$ such that

$$\exists x \in \mathcal{R} - \{0\}, \tilde{\mathbf{F}}\mathbf{X}\tilde{\mathbf{F}}^T = x^2 [\tilde{\mathbf{e}}']_\times \mathbf{X} [\tilde{\mathbf{e}}']_\times^T \tag{33}$$

Now, if we denote $\tilde{\mathbf{F}}\mathbf{X}\tilde{\mathbf{F}}^T$ by $\mathbf{M_F}$ and $[\tilde{\mathbf{e}}']_\times \mathbf{X} [\tilde{\mathbf{e}}']_\times^T$ by $\mathbf{M}_{\mathbf{e}'}$, equation (33) is equivalent to

$$\exists x \in \mathcal{R} - \{0\}, \mathbf{M} = \mathbf{M_F} - x^2 \mathbf{M}'_\mathbf{e} = \mathbf{0}_{3\times 3} \tag{34}$$

Since $\mathbf{M}$ is a symmetric matrix, $\mathbf{M} = \mathbf{0}_{3\times 3}$ is equivalent to a system of six scalar equations of the form $M_{ij} = 0$. Now, since

$$\mathbf{M}\mathbf{e}' = \mathbf{0}_3 \quad \text{where} \quad \mathbf{e}' = \begin{bmatrix} e'_1 & e'_2 & e'_3 \end{bmatrix} \tag{35}$$

this system is in turn equivalent to any system of three equations among which two are coming from a same line i of $\mathbf{M}$, say

$$M_{ii_1} = 0 \quad \text{and} \quad M_{ii_2} = 0 \tag{36}$$

such that $e'_{i_3} \neq 0$, where i_1, i_2 and i_3 are distinct, and the other one from another line $j \neq i$, say

$$M_{jj_1} = 0 \tag{37}$$

such that $e'_{j_2} \neq 0$, where j_1 and j_2 are distinct and different from i. Indeed, equation (35) implies that

$$e'_{i_1} M_{ii_1} + e'_{i_2} M_{ii_2} + e'_{i_3} M_{ii_3} = 0 \tag{38}$$

$$e'_{j_1} M_{jj_1} + e'_{j_2} M_{jj_2} + e'_i M_{ji} = 0 \tag{39}$$

$$e'_1 M_{k1} + e'_2 M_{k2} + e'_3 M_{k3} = 0 \tag{40}$$

where k is the remaining line of $\mathbf{M}$. Equations (38) and (36) then leads to $M_{ii_3} = 0$. Since then $M_{ij} = M_{ji} = 0$, equation (39) and (37) leads to $M_{jj_2} = 0$. Lastly, two equations among $M_{k1} = 0$, $M_{k2} = 0$ and $M_{k3} = 0$ are then satisfied, so that equation (40) leads to the third one. Such a system of three equations always exists since even if only one coordinate of e' is different from zero, say e_l, then one can take $i_3 = j_2 = l$. If all coordinates of e are different from zero, there are 16 such systems. Indeed, one can extract 20 systems of three equations from a system of six equations and, among them, the four following ones are not correct:

$$\{M_{11} = 0, M_{22} = 0, M_{33} = 0\}$$
$$\forall i \in [1,3], \{M_{i1} = 0, M_{i2} = 0, M_{i3} = 0\}$$

So, equation (34) becomes

$$\exists x \in \mathcal{R} - \{0\},$$
$$\mathbf{M}_{\mathbf{F}ii_1} = x^2 \mathbf{M}_{\mathbf{e}'ii_1} \quad \text{and} \quad \mathbf{M}_{\mathbf{F}ii_2} = x^2 \mathbf{M}_{\mathbf{e}'ii_2} \quad \text{and} \quad \mathbf{M}_{\mathbf{F}jj_1} = x^2 \mathbf{M}_{\mathbf{e}'jj_1}$$

which is equivalent to

$$\mathbf{M}_{\mathbf{F}ii_1} \mathbf{M}_{\mathbf{e}'jj_1} = \mathbf{M}_{\mathbf{e}'ii_1} \mathbf{M}_{\mathbf{F}jj_1} \quad \text{and} \quad \mathbf{M}_{\mathbf{F}ii_2} \mathbf{M}_{\mathbf{e}'jj_1} = \mathbf{M}_{\mathbf{e}'ii_2} \mathbf{M}_{\mathbf{F}jj_1} \tag{41}$$

Equations (41) are two homogeneous polynomials, denoted by $P_1(\mathbf{F}, \mathbf{X})$ and $P_2(\mathbf{F}, \mathbf{X})$ of degree two in the six unknown coefficients of

$$\mathbf{X} = \begin{bmatrix} x_1 & x_5 & x_3 \\ x_5 & x_2 & x_4 \\ x_3 & x_4 & x_6 \end{bmatrix}$$

They are the Kruppa equations.

5.3.2 A study of the Kruppa equations

From equation (33), we see that the Kruppa equations are also equivalent to

$$\exists x \in \mathcal{R} - \{0\}, \mathbf{F}\mathbf{X}\mathbf{F}^T = x^2[\mathbf{e}']_\times \mathbf{X}[\mathbf{e}']_\times^T$$

which, according to proposition B.5, is equivalent to

$$\exists x \in \mathcal{R} - \{0\}, \mathbf{X} \in \mathcal{V}_{\mathbf{F},x}$$

where

$$\begin{aligned} \mathcal{V}_{\mathbf{F},\pm 1} &= \mathcal{V}(\mathbf{A}\mathbf{A}^T, \mathbf{A}\mathbf{r}\mathbf{r}^T\mathbf{A}^T, \mathbf{A}\mathbf{S}_{-1}\mathbf{t}\mathbf{t}^T\mathbf{S}_{-1}^T\mathbf{A}^T) \\ \mathcal{V}_{\mathbf{F},x} &= \mathcal{V}(\mathbf{A}\mathbf{S}_x\mathbf{t}\mathbf{t}^T\mathbf{S}_x^T\mathbf{A}^T, \mathbf{A}\mathbf{S}_{-x}\mathbf{t}\mathbf{t}^T\mathbf{S}_{-x}^T\mathbf{A}^T, \mathbf{A}(\mathbf{S}_{x^2}\mathbf{t}\mathbf{r}^T + \mathbf{r}\mathbf{t}^T\mathbf{S}_{x^2}^T)\mathbf{A}^T) \end{aligned}$$

if $\mathbf{t}$ is not proportional to $\mathbf{r}$ or

$$\begin{aligned} \mathcal{V}_{\mathbf{F},\pm 1} &= \mathcal{V}(\mathbf{A}\mathbf{A}^T, \mathbf{A}\mathbf{r}\mathbf{r}^T\mathbf{A}^T) \\ \mathcal{V}_{\mathbf{F},x} &= \mathcal{V}(\mathbf{A}\mathbf{r}\mathbf{r}^T\mathbf{A}^T) \end{aligned}$$

otherwise and with

$$\mathbf{S}_\lambda = \mathbf{R}^2 + (\lambda - \mathrm{tr}(\mathbf{R}))\mathbf{R} + (\lambda^2 - \mathrm{tr}(\mathbf{R})\lambda + \mathrm{tr}(\mathbf{R}))\mathbf{I}_3$$

Appendix A is devoted to the study of $\mathbf{S}_\lambda$ which helps in demonstrating the above result in appendix B.

This result shows that $\mathbf{K}$ belongs to a solution vector space and so, that one set of two Kruppa equations (41) is not enough to compute $\mathbf{K}$. Appendix B, together with appendices C and D, also studies how a system of two sets of two Kruppa equations constrain $\mathbf{K}$. More precisely, proposition B.9 shows that such a system obtained, for example, from three views with the same intrinsic parameters $\mathbf{A}$, is, except for particular spatial configurations of the views that are detailed in table 9, in general enough for $\mathbf{K}$ to be an isolated solution of the system, that is such that there exists a neighborhood around $\mathbf{K}$ that contains only solutions proportional to $\mathbf{K}$.

5.3.3 Solving the Kruppa equations

In practice, more than three views are available, so that $\mathbf{K}$ is computed as the solution of a non-linear least squares problem [1, 15].

More precisely, $\mathbf{K}$ is computed as the matrix that minimizes the following criterion:

$$\mathcal{C}(\mathbf{X}) = \sum_{i=1}^{N} \left(\frac{P_1^2(\mathbf{F}_i, \mathbf{X})}{\frac{\partial P_1}{\partial \mathbf{F}}^T(\mathbf{F}_i, \mathbf{X}) \mathbf{\Lambda}_{\mathbf{F}_i} \frac{\partial P_1}{\partial \mathbf{F}}(\mathbf{F}_i, \mathbf{X})} + \frac{P_2^2(\mathbf{F}_i, \mathbf{X})}{\frac{\partial P_2}{\partial \mathbf{F}}^T(\mathbf{F}_i, \mathbf{X}) \mathbf{\Lambda}_{\mathbf{F}_i} \frac{\partial P_2}{\partial \mathbf{F}}(\mathbf{F}_i, \mathbf{X})} \right) \tag{42}$$

where N is the number of fundamental matrices $\mathbf{F}_i$ available. If n views are available, we have $N = \frac{n(n-1)}{2}$. The minimization is done using a classical Levenberg-Marquardt method [1, 15].

$\mathbf{X}$ is parametrized by an upper triangular matrix $\mathbf{U}$ such that $\mathbf{X} = \mathbf{U}\mathbf{U}^T$, in order to be a symmetric positive definite matrix. Moreover, the bottom-right element of $\mathbf{U}$ is fixed to 1, which amounts to take $x_6 = 1$: This eliminates the solutions proportional to $\mathbf{K}$. The minimization is thus done over five parameters.

Three ways of initialization are possible:

- Taking three fundamental matrices leads to a system of six polynomials in five unknowns. From this system, we build six systems of five polynomials and solve them using a homotopic method.
- Assuming that $\gamma = 0$ (see equation (1)), which is a good approximation for usual cameras, allows to introduce the following additional polynomial equation:

  $$x_5 = x_3 x_4$$

  so that taking two fundamental matrices leads to a system of five polynomials in five unknowns which is also solved using a homotopic method.
- Assuming that $\gamma = 0$ and (u_0, v_0) is the center of the image, which is also a good approximation for usual cameras, leaves only x_1 and x_2 as unknowns so that taking one fundamental matrix leads to a system of two polynomials of degree two in two unknowns which is solved analytically.

In each of these three ways, several systems are extracted from several fundamental matrices and the solution common to all of them is chosen. The more systems are taken in account, the more precise is the solution. Since, on one hand, they generate a lot of solutions and, on the other hand, they need the use of a homotopic method, which is a quite fastidious process, the systems built by the first two initializations are not well-adapted to a large number of images. On the contrary, the third initialization allows a quick statistical processing: The points $(\alpha_u = \sqrt{x_1}, \alpha_v = \sqrt{x_2})$ are distributed along a line whose slope gives the aspect ratio $\delta = \frac{\alpha_v}{\alpha_u}$; this line is estimated by robust fitting and the corresponding outliers are not used in the process of minimization; the mean is taken as initial value. Figures 3, 4 and 5 show that choosing (u_0, v_0) around the image center does not have a big influence on the computed value of δ.

5.4 Using geometric information

If we have some a priori information about the scene such as N_a angles θ_j between lines and N_r ratios ρ_j of lengths, equations (24) and (25) are used to add to the criterion giving by equation (42) the following criteria:

$$\mathcal{C}_{angle}(\mathbf{X}) = \sum_{j=1}^{N_a}\sum_{i=1}^{N} \frac{(\cos(\theta_j) - C_j(\mathbf{F}_i, \mathbf{X}))^2}{\frac{\partial C_j}{\partial \mathbf{F}}^T(\mathbf{F}_i, \mathbf{X})\mathbf{\Lambda}_{\mathbf{F}_i}\frac{\partial C_j}{\partial \mathbf{F}}(\mathbf{F}_i, \mathbf{X})}$$
$$\mathcal{C}_{ratio}(\mathbf{X}) = \sum_{j=1}^{N_r}\sum_{i=1}^{N} \frac{(\rho_j - R_j(\mathbf{F}_i, \mathbf{X}))^2}{\frac{\partial R_j}{\partial \mathbf{F}}^T(\mathbf{F}_i, \mathbf{X})\mathbf{\Lambda}_{\mathbf{F}_i}\frac{\partial R_j}{\partial \mathbf{F}}(\mathbf{F}_i, \mathbf{X})}$$

where

$$C_j(\mathbf{F}_i, \mathbf{X}) = c(\mathbf{v}_{1ij}, \mathbf{v}_{2ij}) \quad \text{and} \quad R_j(\mathbf{F}_i, \mathbf{X}) = r(\mathbf{v}_{abij}, \mathbf{v}_{acij}, \mathbf{v}_{bcij}, \mathbf{v}_{bdij}, \mathbf{v}_{cdij})$$

In order to evaluate $\mathcal{C}_{angle}(\mathbf{A}\mathbf{A}^T)$ and $\mathcal{C}_{ratio}(\mathbf{A}\mathbf{A}^T)$, given a current $\mathbf{A}$ at each iteration of the processus of minimization, and since these criteria require computing the images of points at infinity, we need to compute $\mathbf{H}_\infty$ from $\mathbf{F}$ and $\mathbf{A}$.

For that, we first compute $\mathbf{E}$, using equation (10), and then, decompose $\mathbf{E}$ like in equation (11) to recover $\mathbf{R}$, $\mathbf{H}_\infty$ being then given by equation (5). In [6], Faugeras shows that the decomposition of $\mathbf{E}$ like in equation (11) is possible if and only if $\mathbf{E}$ has one singular value equal to zero and the other two singular values equal to each other. Now, applying equation (10) directly leads to a matrix $\tilde{\mathbf{E}} = \mathbf{A}^T\mathbf{F}\mathbf{A}$ which does have a singular value equal to zero, since $\mathbf{F}$ is parametrized in order to be exactly singular, but whose two other singular values r and s are not necessarily equal. In [10], Hartley explains that the closest matrix $\mathbf{E}$ to $\tilde{\mathbf{E}}$ under the sum-of-squares norm which decomposes like in equation (11) is obtained by replacing r and s by $\frac{r+s}{2}$ in the singular value decomposition of $\tilde{\mathbf{E}}$. Now, there are two ways to decompose $\mathbf{E}$ like in equation (11) [10]. Indeed, if we denote $\mathbf{R_t}$ the rotation matrix of axis $\mathbf{t}$ and angle π, we have[3]

$$\mathbf{R_t} = \mathbf{I}_3 + 2\frac{[\mathbf{t}]_\times^2}{\|\mathbf{t}\|^2}$$

[3] using the Rodrigues formula $\mathbf{R} = \mathbf{I}_3 + \sin(\theta)[\mathbf{r}]_\times + (1 - \cos(\theta))[\mathbf{r}]_\times^2$

so that[4]

$$\begin{aligned}\mathbf{E} &= [-\mathbf{t}]_\times \mathbf{R_t}\mathbf{R} \\ &= [-\mathbf{t}]_\times \mathbf{R} - 2\frac{[\mathbf{t}]_\times^3}{\|\mathbf{t}\|^2}\mathbf{R} \\ &= [\mathbf{t}]_\times \mathbf{R}\end{aligned}$$

To compute these two rotation matrices, we first compute a vector proportionnal to $\mathbf{t}$, according to equations (31) and (6):

$$\tilde{\mathbf{t}} = \mathbf{A}^{-1}\tilde{\mathbf{e}}' = \mu\mathbf{t}$$

where μ is an unknown scalar. $\mathbf{R}$ and $\mathbf{R_t}\mathbf{R}$ are then equal to

$$\frac{1}{\|\tilde{\mathbf{t}}\|^2}(\mathbf{E}^{*T} \pm [\tilde{\mathbf{t}}]_\times \mathbf{E})$$

where $\mathbf{E}^*$ is the matrix of cofactors of $\mathbf{E}$ [6]. When dealing with images that come from a sequence, we choose the rotation of smallest angle.

6 Computing the projection matrices from A and F

The n views considered in this section are assumed to have the same intrinsic parameters matrix $\mathbf{A}$ and are such that the fundamental matrices between each two views are known.

For any view i of optical center C_i and any view j of optical center C_j, we denote

$$\mathbf{Q}_{\mathcal{F}_{C_i}}^{\mathcal{F}_{C_j}} = \begin{bmatrix}\mathbf{R}_{ij} & \mathbf{t}_{ij} \\ \mathbf{0}_3^T & 1\end{bmatrix}$$

According to section 5.4, we can compute $\tilde{\mathbf{t}}_{ij} = \mu_{ij}\mathbf{t}_{ij}$ and $\mathbf{R}_{ij}$ where μ_{ij} is an unknown scalar. We then define the frame $\mathcal{F}_E$ by its matrix of change of frame from $\mathcal{F}_{C_1}$ to $\mathcal{F}_E$:

$$\mathbf{Q}_{\mathcal{F}_{C_1}}^{\mathcal{F}_E} = \begin{bmatrix}\mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3^T & \frac{1}{\mu_{12}}\end{bmatrix} \tag{43}$$

[4]using the algebraic equation $[\mathbf{u}]_\times^3 = -\|\mathbf{u}\|^2[\mathbf{u}]_\times$

In $\mathcal{F}_E$, we have then

$$\begin{aligned} \mathbf{P}_1 &= \mathbf{A}\mathbf{P}_0\mathbf{Q}_{\mathcal{F}_E}^{\mathcal{F}_{C_1}} = \mathbf{A}\begin{bmatrix}\mathbf{I}_3 & \mathbf{0}_3\end{bmatrix} \\ \mathbf{P}_2 &= \mathbf{A}\mathbf{P}_0\mathbf{Q}_{\mathcal{F}_E}^{\mathcal{F}_{C_2}} = \mathbf{A}\mathbf{P}_0\mathbf{Q}_{\mathcal{F}_{C_1}}^{\mathcal{F}_{C_2}}\mathbf{Q}_{\mathcal{F}_E}^{\mathcal{F}_{C_1}} = \mathbf{A}\begin{bmatrix}\mathbf{R}_{12} & \tilde{\mathbf{t}}_{12}\end{bmatrix} \end{aligned}$$

and for any view $i > 2$

$$\mathbf{P}_i = \mathbf{A}\mathbf{P}_0\mathbf{Q}_{\mathcal{F}_E}^{\mathcal{F}_{C_i}} = \mathbf{A}\mathbf{P}_0\mathbf{Q}_{\mathcal{F}_{C_1}}^{\mathcal{F}_{C_i}}\mathbf{Q}_{\mathcal{F}_E}^{\mathcal{F}_{C_1}} = \mathbf{A}\begin{bmatrix}\mathbf{R}_{1i} & \frac{\mu_{12}}{\mu_{1i}}\tilde{\mathbf{t}}_{1i}\end{bmatrix}$$

Now from

$$\mathbf{Q}_{\mathcal{F}_{C_1}}^{\mathcal{F}_{C_i}} = \mathbf{Q}_{\mathcal{F}_{C_2}}^{\mathcal{F}_{C_i}}\mathbf{Q}_{\mathcal{F}_{C_1}}^{\mathcal{F}_{C_2}}$$

we deduce that

$$\mathbf{t}_{1i} = \mathbf{R}_{2i}\mathbf{t}_{12} + \mathbf{t}_{2i}$$

so that

$$\frac{\mu_{12}}{\mu_{1i}} = \frac{(\tilde{\mathbf{t}}_{2i} \times \tilde{\mathbf{t}}_{1i})^T(\tilde{\mathbf{t}}_{2i} \times \mathbf{R}_{2i}\tilde{\mathbf{t}}_{12})}{\|\tilde{\mathbf{t}}_{2i} \times \tilde{\mathbf{t}}_{1i}\|^2}$$

The projection matrices computed this way are in practice refined using a technique similar to the classical bundle adjustment method.

For that, we assume that we have p point correspondences corresponding to p points M_i of the scene of unknowns coordinates $(\mathbf{M}_i)_{i\in[1,p]}$. The refinement of the projections matrices is then done by minimizing the following criterion:

$$\sum_{j=1}^{p}\sum_{i\in I_j}((x_{ij} - \frac{\mathbf{l}_{i1}^T\mathbf{M}_j}{\mathbf{l}_{i3}^T\mathbf{M}_j})^2 + (y_{ij} - \frac{\mathbf{l}_{i2}^T\mathbf{M}_j}{\mathbf{l}_{i3}^T\mathbf{M}_j})^2) \tag{44}$$

over

- the set of all the points coordinates $\mathbf{M}_i$ (for each j, I_j is the set of the indices i of the images in which the projection $\mathbf{m}_{ij} = \begin{bmatrix}x_{ij} & y_{ij}\end{bmatrix}^T$ of M_j is known),
- the set of the projection matrices of the n views $(\mathbf{P}_i)_{i\in[1,n]}$ so that

$$\forall i \in [1, n], \mathbf{P}_i = \begin{bmatrix}\mathbf{l}_{i1}^T \\ \mathbf{l}_{i2}^T \\ \mathbf{l}_{i3}^T\end{bmatrix}$$

More precisely, this is done by minimizing the criterion (44) over the set of the projection matrices only, by using a classical Levenberg-Marquardt method : at each evaluation of the criterion, the points coordinates $\mathbf{M}_j$ are linearly computed from the equation

$$\forall i \in I_j, \mathbf{m}_{ij} \times \mathbf{P}_i \mathbf{M}_j = \mathbf{0}_3$$

The projection matrices are parametrized the following way

$$\mathbf{P}_1 = \mathbf{A}\begin{bmatrix}\mathbf{I}_3 & \mathbf{0}_3\end{bmatrix} \quad \text{and} \quad \forall i \in [2, n], \mathbf{P}_i = \mathbf{A}\begin{bmatrix}\mathbf{R}_i & \mathbf{t}_i\end{bmatrix}$$

which makes a total of $5 + 6(n-1)$ parameters. $\mathbf{R}_i$ is initialized by $\mathbf{R}_{1i}$ and $\mathbf{t}_i$ by $\frac{\mu_{12}}{\mu_{1i}}\mathbf{t}_{1i}$.

7 Results

In this section, we show the results obtained with three real sequences. The point correspondences have been obtained automatically by tracking, in the case of the sequence 1 (figure 2) and manually, in the case of the sequences 2 (figure 7) and 3 (figure 11).

For each of the sequence, the results concerning the self-calibration technique described in section 5.3.3 are first shown. Three kinds of minimization are tested depending on the parametrization of $\mathbf{A}$: five parameters, four parameters ($\gamma = 0$), four parameters ($\alpha_v = \delta\alpha_u$), three parameters ($\gamma = 0$ and $\alpha_v = \delta\alpha_u$), where δ is the aspect ratio evaluated at the initialization. In the case of the sequence 1, the intrinsic parameters computed with a classical calibration method[9] are also given. The aspect ratios are very similar. The other parameters are also well estimated, above all with the two last kinds of minimizations, considering the fact that, even with classical calibration methods, they are estimated up to 5% for α_u and α_v and 10% for u_0 and v_0.

Metric measurements, computed directly in the images, using the equations given in section 3.2.3, are then shown and compared with real measurements. Each measurement shown is the median of all the measurements computed from all the pairs of images of the sequence.

At last, for the sequences 2 and 3, a reconstruction is shown that has been obtained after computing the projection matrices by the method described in section 6. Metric measurements, computed from the reconstructed points, are also shown.

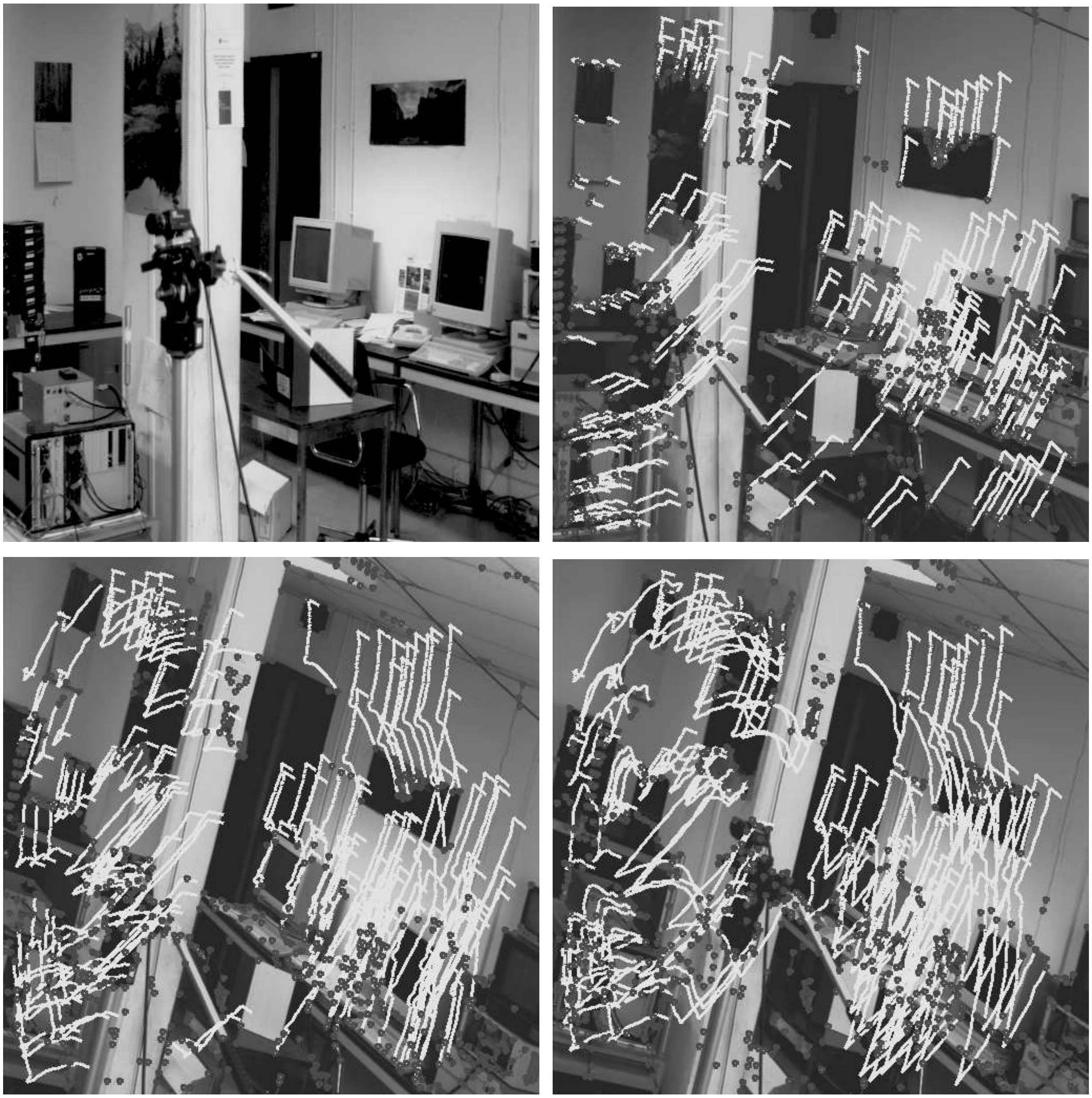


Figure 2: **Sequence 1** : 43 512×512-views; the tracks of the points in four images (1, 12, 28, 43).

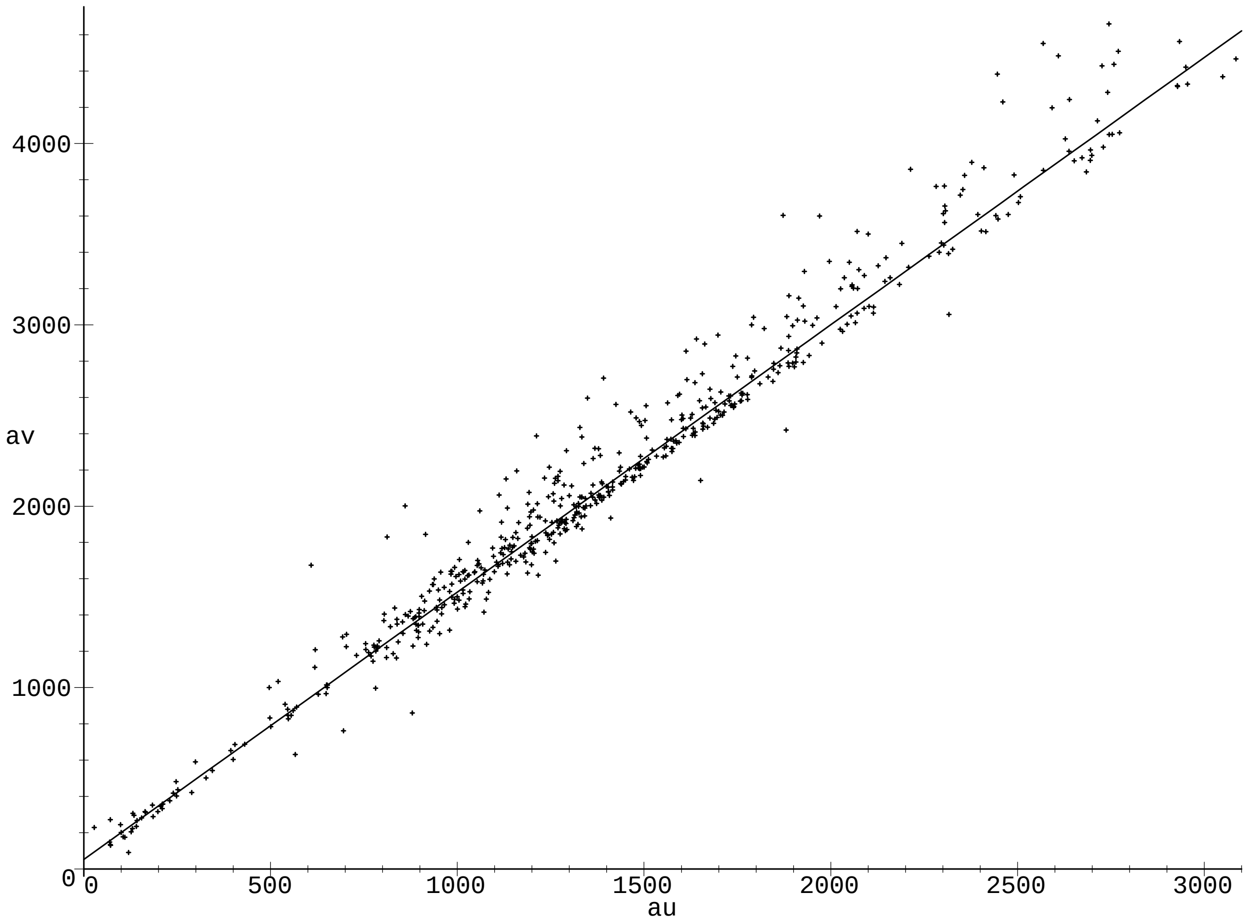


Figure 3: **Sequence 1** : Robust fit of the initial values of α_u and α_v; the slope of the line is the aspect ratio; here: $u_0 = 256$, $v_0 = 256$ and $\delta = 1.474$.

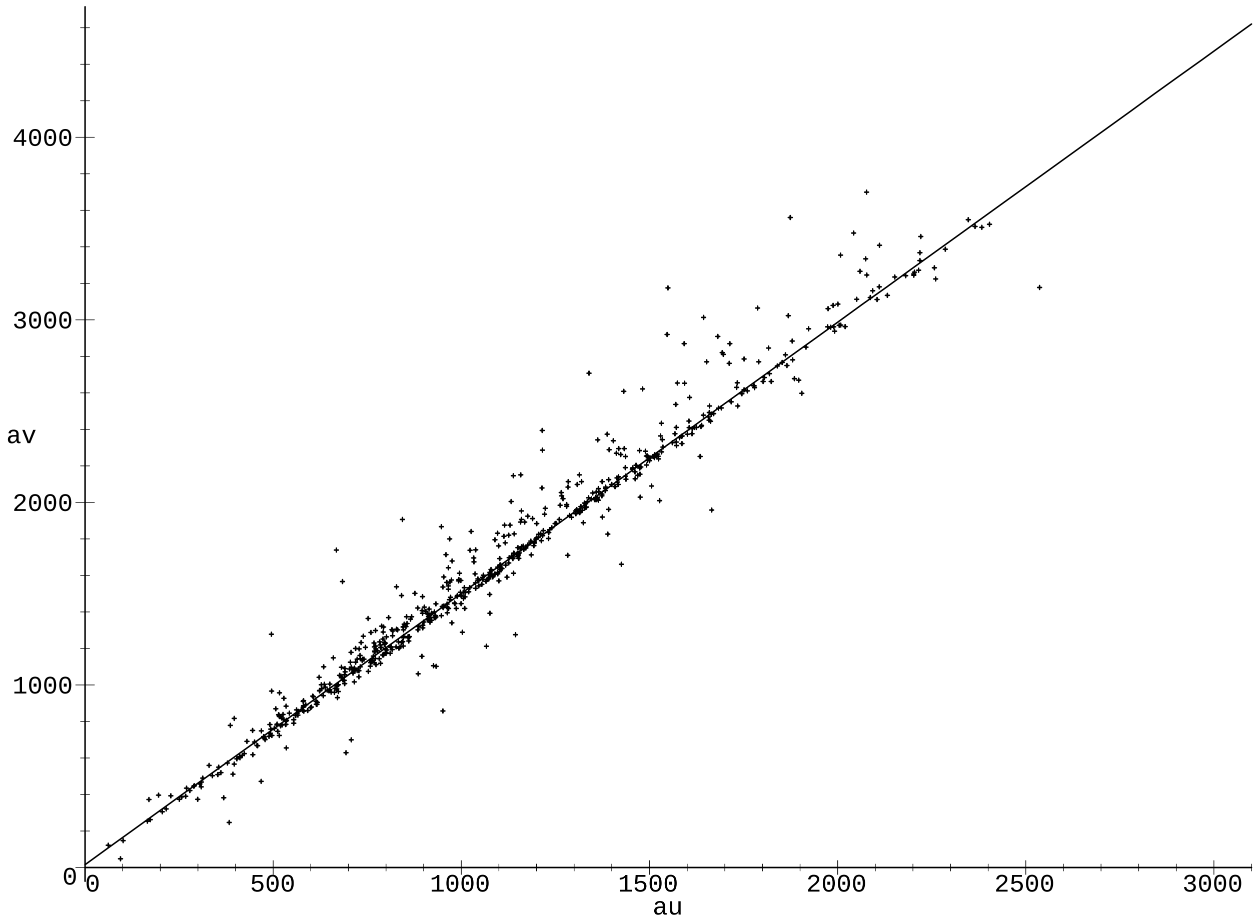


Figure 4: **Sequence 1** : Robust fit of the initial values of α_u and α_v; the slope of the line is the aspect ratio; here: $u_0 = 200$, $v_0 = 300$ and $\delta = 1.485$.

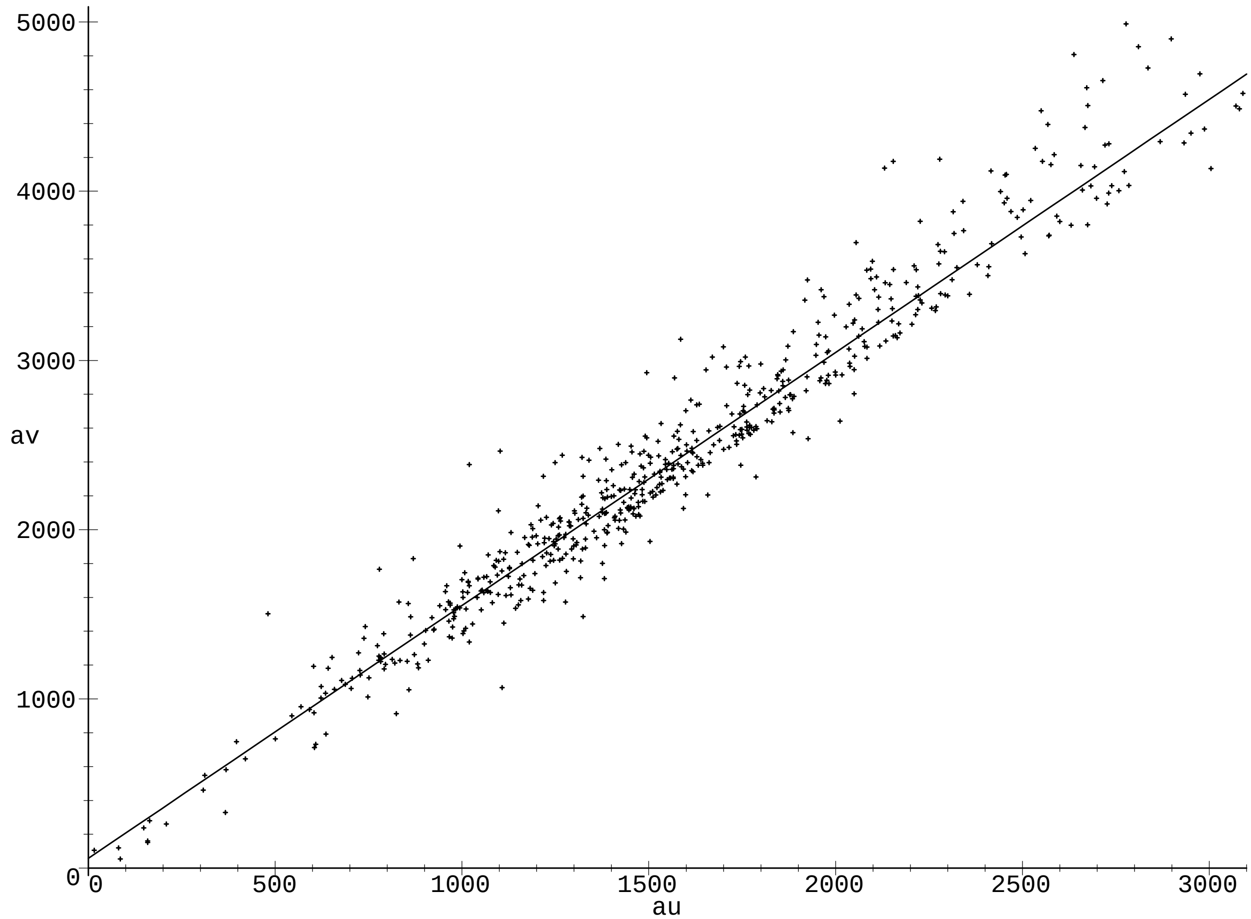


Figure 5: **Sequence 1** : Robust fit of the initial values of α_u and α_v; the slope of the line is the aspect ratio; here: $u_0 = 300$, $v_0 = 200$ and $\delta = 1.495$.

Minimization over 5 parameters	
$\mathbf{A} = \begin{bmatrix} 1164.15 & -9.96 & 198.34 \\ 0 & 1761.47 & 289.5 \\ 0 & 0 & 1 \end{bmatrix}$	$\delta = 1.513$
Minimization over 4 parameters : $\gamma = 0$	
$\mathbf{A} = \begin{bmatrix} 1094.4 & 0 & 208.72 \\ 0 & 1649 & 269.2 \\ 0 & 0 & 1 \end{bmatrix}$	$\delta = 1.508$
Minimization over 4 parameters : $\beta = \delta_0 \alpha$	
$\mathbf{A} = \begin{bmatrix} 952.66 & -0.36 & 215.26 \\ 0 & 1404.07 & 214.3 \\ 0 & 0 & 1 \end{bmatrix}$	$\delta = 1.474$
Minimization over 3 parameters : $\gamma = 0$ et $\beta = \delta_0 \alpha$	
$\mathbf{A} = \begin{bmatrix} 949.82 & 0 & 215.26 \\ 0 & 1399.88 & 214.3 \\ 0 & 0 & 1 \end{bmatrix}$	$\delta = 1.474$
Classical calibration	
$\mathbf{A} = \begin{bmatrix} 940 & 0.05 & 249.35 \\ 0 & 1387 & 255.35 \\ 0 & 0 & 1 \end{bmatrix}$	$\delta = 1.475$

Table 1: **Sequence 1** : The intrinsic parameters computed by self-calibration compared with those obtained using a calibration pattern.

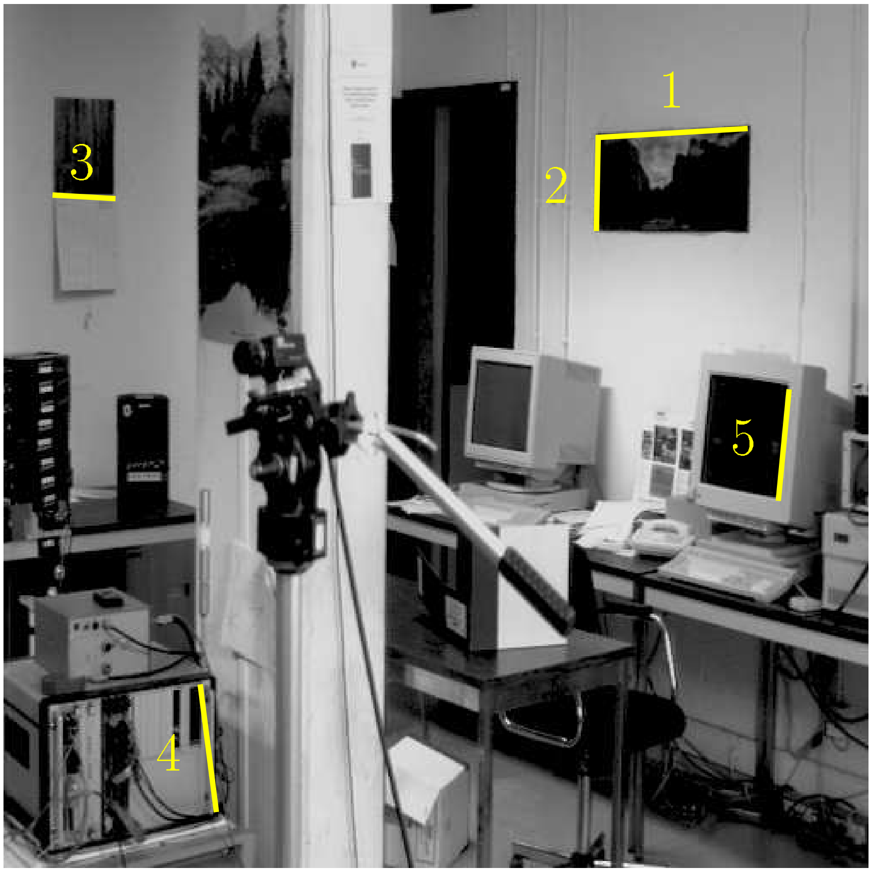


Figure 6: **Sequence 1** : The segments used for measurements of table 2.

Angle	Computed	Real	Ratio	Computed	Real
cos(1, 2)	0.079	0.089	1/2	2.68	2.8
cos(1, 3)	0.079	0.029	1/3	2.9	2.98
cos(2, 4)	0.91	0.94	2/4	1.1	1.2
cos(3, 5)	0.4	0.34	3/5	1.16	1.05

Table 2: **Sequence 1** : Some measurements relative to the segments of figure 6, computed directly in the images.

Figure 7: **Sequence 2** : 10 768×512-views of the Arcades place in Valbonne (1, 2, 4, 6, 8, 10).

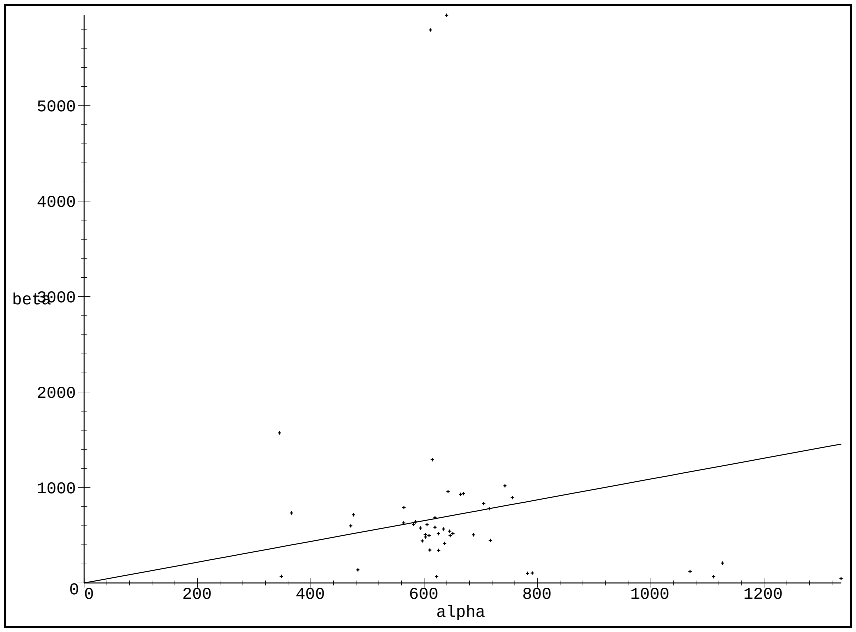
Initialisation

$\delta_0 = 1.089$
Median
$\mathbf{A} = \begin{bmatrix} 625.061 & 0 & 384 \\ 0 & 564.979 & 256 \\ 0 & 0 & 1 \end{bmatrix}$ $\delta = 0.904$
Minimization over 5 parameters
$\mathbf{A} = \begin{bmatrix} 667.732 & 9.832 & 415.015 \\ 0 & 717.939 & 328.047 \\ 0 & 0 & 1 \end{bmatrix}$ $\delta = 1.075$
Minimization over 4 parameters : $\gamma = 0$
$\mathbf{A} = \begin{bmatrix} 670.747 & 0 & 416.072 \\ 0 & 729.394 & 334.974 \\ 0 & 0 & 1 \end{bmatrix}$ $\delta = 1.087$
Minimization over 4 parameters : $\beta = \delta_0 \alpha$
$\mathbf{A} = \begin{bmatrix} 666.705 & 9.897 & 414.542 \\ 0 & 726.272 & 327.606 \\ 0 & 0 & 1 \end{bmatrix}$ $\delta = 1.089$
Minimization over 3 parameters : $\gamma = 0$ et $\beta = \delta_0 \alpha$
$\mathbf{A} = \begin{bmatrix} 670.566 & 0 & 415.958 \\ 0 & 730.478 & 327.606 \\ 0 & 0 & 1 \end{bmatrix}$ $\delta = 1.089$

Table 3: **Sequence 2** : The intrinsic parameters computed by self-calibration.

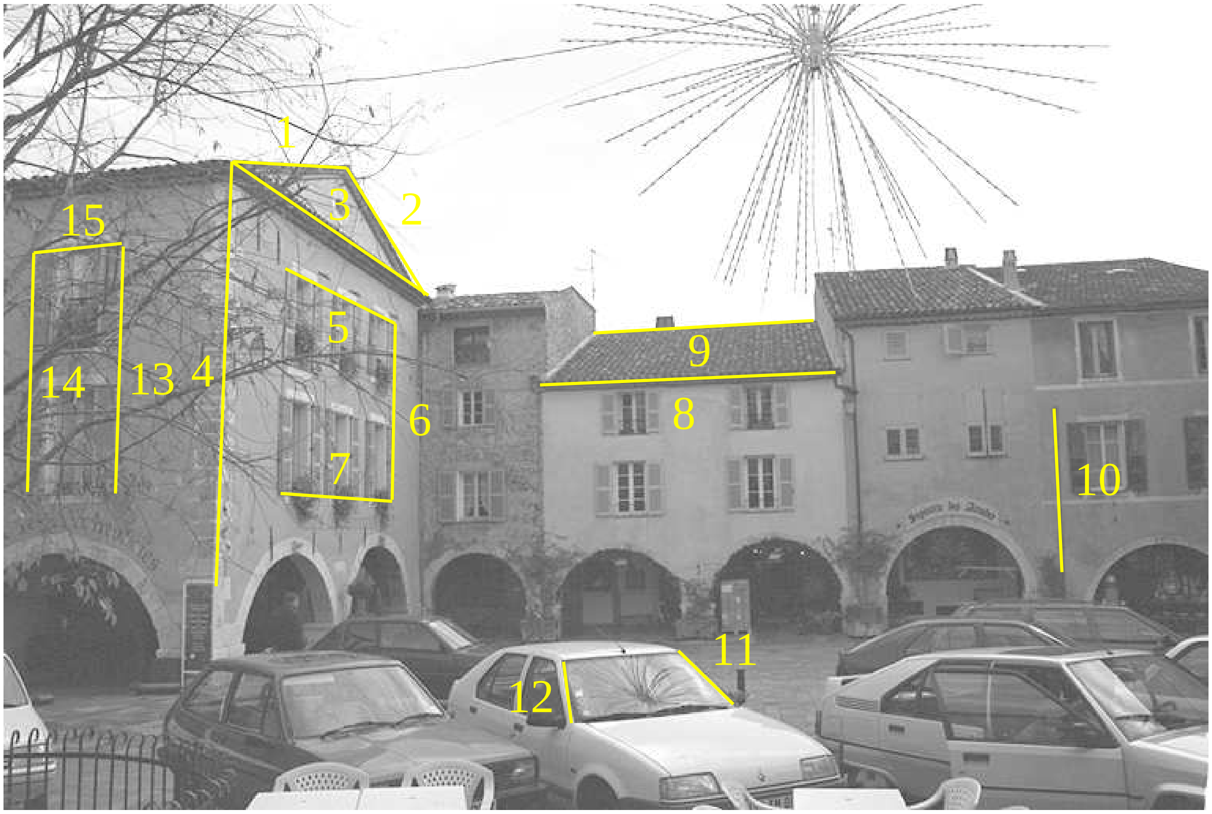


Figure 8: **Sequence 2** : The segments used for the measurements of table 4.

Angle	Computed	Real	Ratio	Computed	Real
$\cos(3,4)$	0.031	0	1/2	1.04	1
$\cos(4,8)$	0.105	0	5/7	0.983	1
$\cos(8,10)$	0.032	0	11/12	0.373	1
$\cos(5,6)$	0.031	0	13/14	0.994	1
$\cos(13,15)$	0.07	0			

Table 4: **Sequence 2** : Some measurements relative to the segments of figure 8, computed directly in the images.

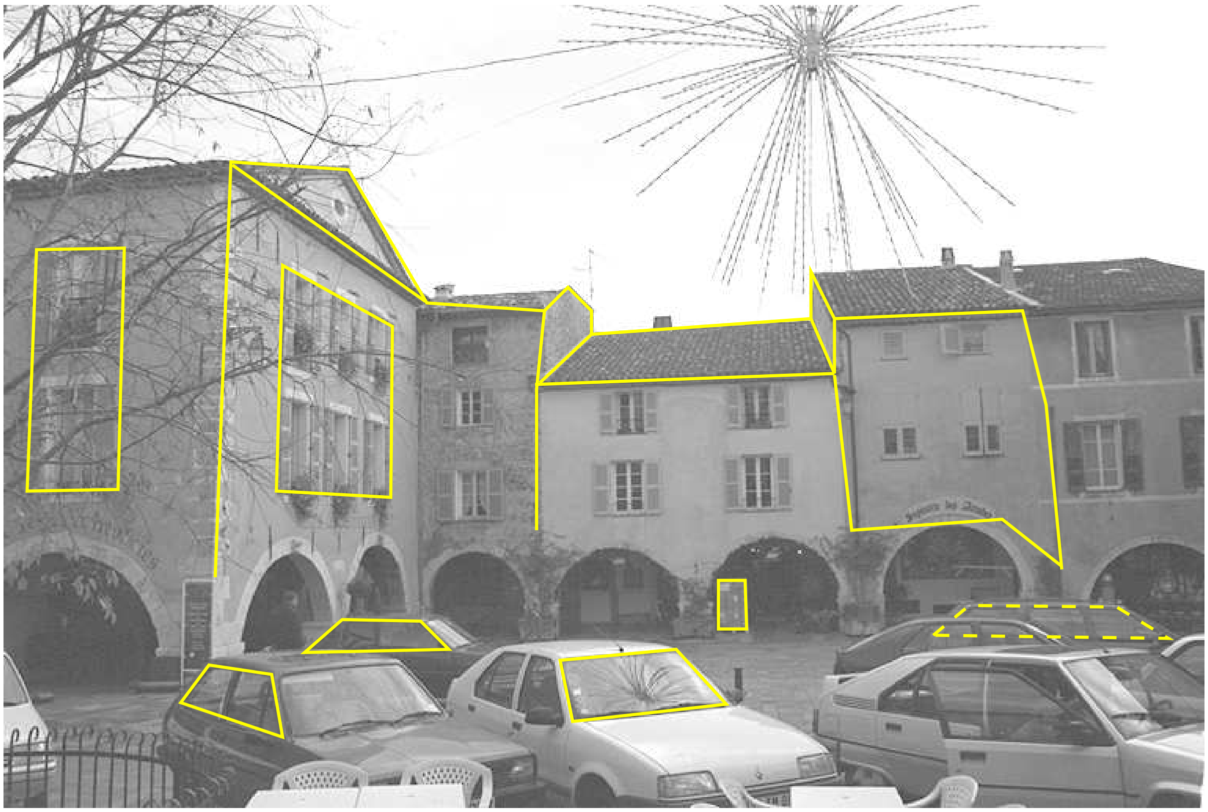


Figure 9: **Sequence 2** : The segments reconstructed in figure 10.

Angle	Computed	Real	Ratio	Computed	Real
$\cos(3,4)$	0.017	0	1/2	1.01	1
$\cos(4,8)$	0.004	0	5/7	0.954	1
$\cos(8,10)$	0.028	0	11/12	0.987	1
$\cos(5,6)$	0.007	0	13/14	0.983	1
$\cos(13,15)$	0.062	0			

Table 5: **Sequence 2** : Some measurements relative to the segments of figure 8, computed from the reconstructed points.

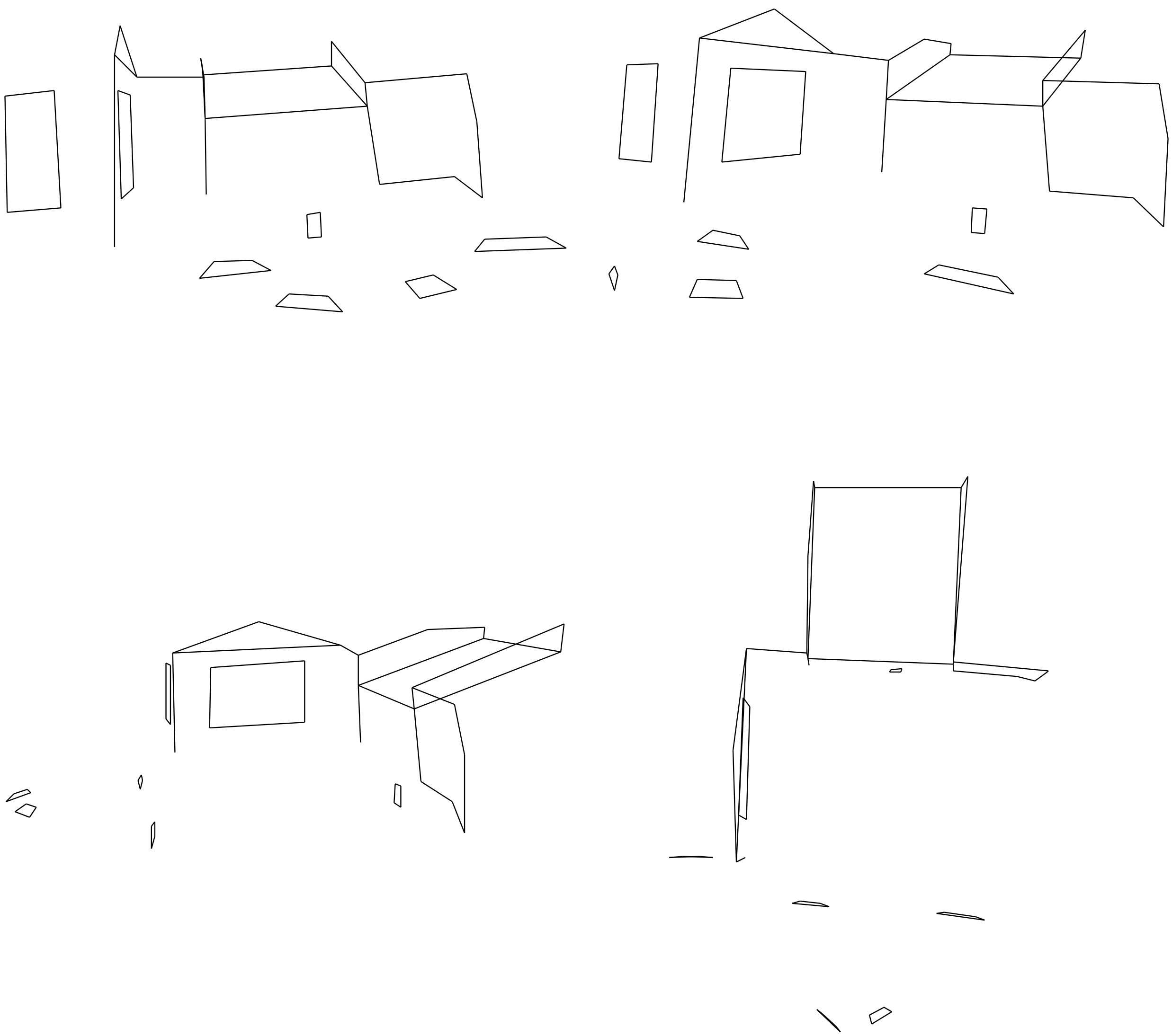


Figure 10: **Sequence 2** : Four views of the reconstruction of the segments of figure 9.

Figure 11: **Sequence 3** : 15 512×768-views of the church of Valbonne (1, 3, 5, 10, 12, 14).

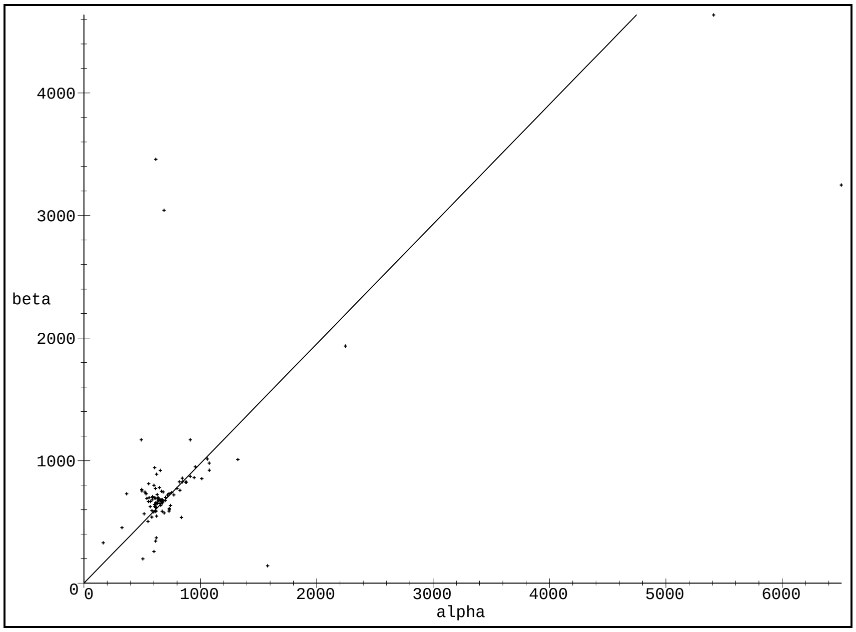
Initialisation

$\delta_0 = 0.977$
Median
$\mathbf{A} = \begin{bmatrix} 639.460 & 0 & 256 \\ 0 & 693.682 & 384 \\ 0 & 0 & 1 \end{bmatrix}$ $\delta = 1.085$
Minimization over 5 parameters
$\mathbf{A} = \begin{bmatrix} 679.515 & -3.258 & 254.466 \\ 0 & 677.470 & 383.476 \\ 0 & 0 & 1 \end{bmatrix}$ $\delta = 0.997$
Minimization over 4 parameters : $\gamma = 0$
$\mathbf{A} = \begin{bmatrix} 681.345 & 0 & 258.802 \\ 0 & 679.285 & 383.188 \\ 0 & 0 & 1 \end{bmatrix}$ $\delta = 0.997$
Minimization over 4 parameters : $\beta = \delta_0\alpha$
$\mathbf{A} = \begin{bmatrix} 694.213 & -3.578 & 249.650 \\ 0 & 677.954 & 377.627 \\ 0 & 0 & 1 \end{bmatrix}$ $\delta = 0.977$
Minimization over 3 parameters : $\gamma = 0$ et $\beta = \delta_0\alpha$
$\mathbf{A} = \begin{bmatrix} 694.998 & 0 & 255.448 \\ 0 & 678.721 & 377.627 \\ 0 & 0 & 1 \end{bmatrix}$ $\delta = 0.977$

Table 3: **Sequence 3** : The intrinsic parameters computed by self-calibration.

Figure 12: **Sequence 3** : The segments used for the measurements of table 7.

Angle	Computed	Real	Ratio	Computed	Real
$\cos(1,3)$	1	1	2/4	0.993	1
$\cos(1,2)$	0.054	0	4/5	0.999	1
$\cos(3,5)$	0.032	0	10/12	1.05	1
$\cos(6,7)$	0.999	1	8/9	1.004	1
$\cos(15,16)$	1	1	17/18	0.966	1
$\cos(19,20)$	0.101	0	13/14	0.902	1

Table 7: **Sequence 3** : Some measurements relative to the segments of figure 12, computed directly in the images.

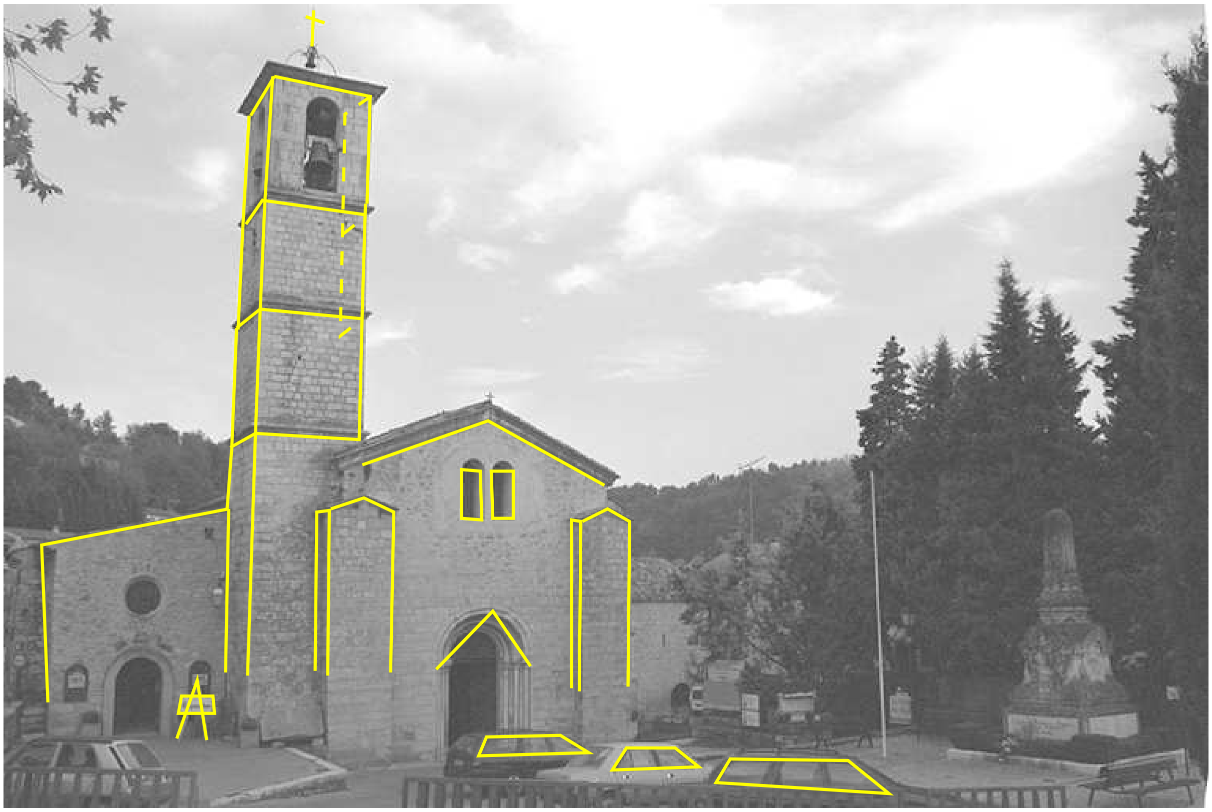


Figure 13: **Sequence 3** : The segments reconstructed in figure 14.

Angle	Computed	Real	Ratio	Computed	Real
cos(1,3)	1	1	2/4	0.982	1
cos(1,2)	0.026	0	4/5	0.999	1
cos(3,5)	0.018	0	10/12	1.022	1
cos(6,7)	1	1	8/9	1.007	1
cos(15,16)	1	1	17/18	0.984	1
cos(19,20)	0.022	0	13/14	0.955	1

Table 8: **Sequence 3** : Some measurements relative to the segments of figure 12, computed from the reconstructed points.

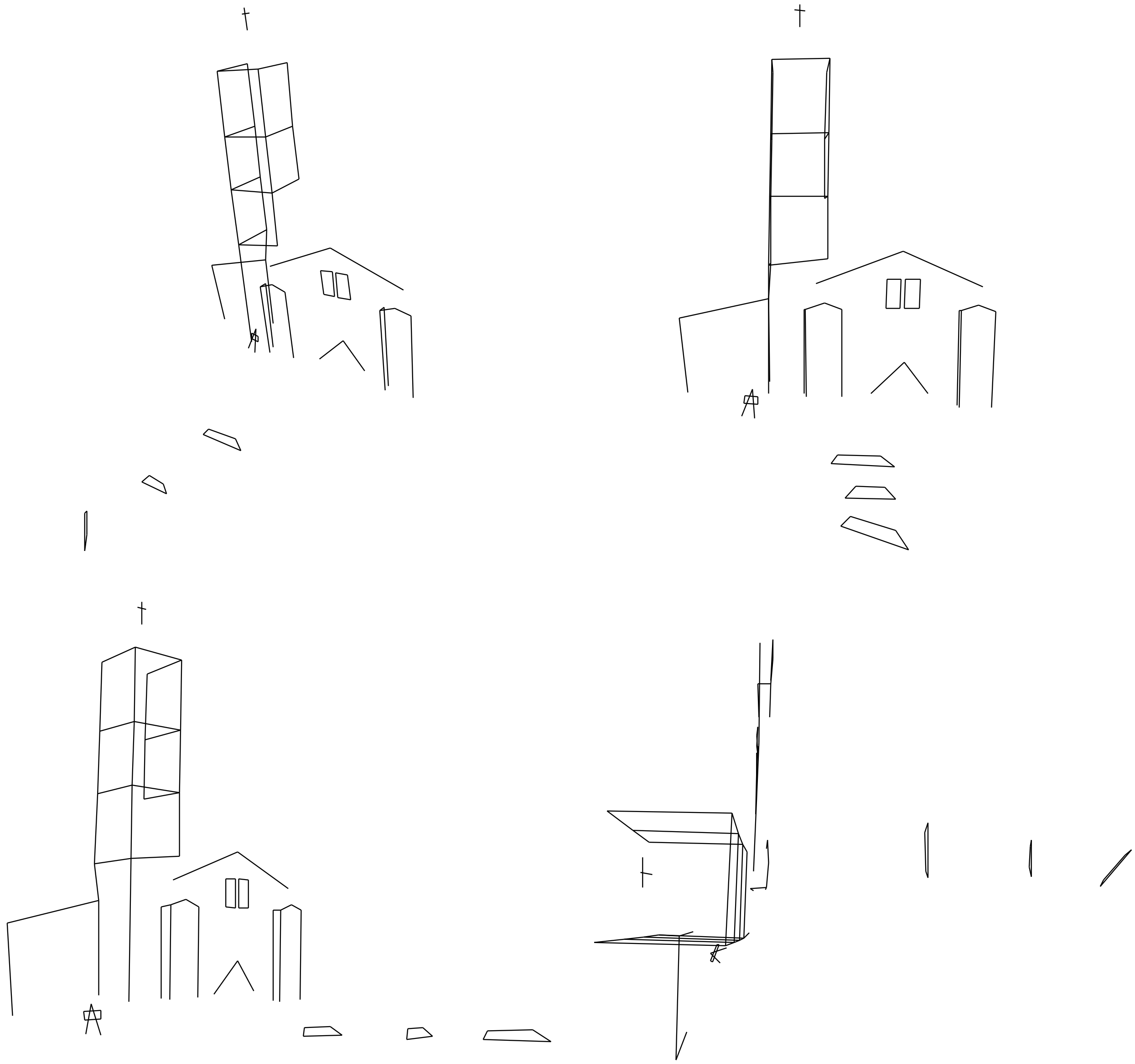


Figure 14: **Sequence 3** : Four views of the reconstruction of the segments of figure 13.

8 Conclusion

The self-calibration technique described in this article is the generalization to a large number of images of the algorithm developped by Luong and Faugeras[8]. In comparison with the original, its robustness is improved essentially because of the use of statistical tools, such as robust model fitting and covariance matrices, at each step of the process (initialization and minimization).

Its main advantages over other algorithms[11] are that its initialization is easy and needs no a priori knowledge about the camera due to the algebraic structure of the equations and that the number of parameters over which the minimization is done is low which makes it very efficient.

References

[1] E. Anderson, Z. Bai, C. Bishof, J. Demmel, J. Dongarra, J. Du Croz, A. Greenbaum, S. Hammarling, A. McKenney, S. Ostrouchov, and D. Sorensen. *LAPACK Users' Guide.* Society for Industrial and Applied Mathematics, 3600 University City Science Center, Philadelphia, PA 19104-2688, second edition, 1994.

[2] Boubakeur Boufama, Roger Mohr, and Françoise Veillon. Euclidean Constraints for Uncalibrated Reconstruction. Technical Report RT96 IMAG 17 LIFIA, LIFIA, INSTITUT IMAG, March 1993.

[3] Gabriella Csurka, Cyril Zeller, Zhengyou Zhang, and Olivier Faugeras. Characterizing the uncertainty of the fundamental matrix. Technical Report 2560, INRIA, 1995.

[4] O.D. Faugeras and F. Lustman. Let us suppose that the world is piecewise planar. In O. D. Faugeras and Georges Giralt, editors, *Robotics Research, The Third International Symposium*, pages 33–40. MIT Press, 1986.

[5] Olivier Faugeras. What can be seen in three dimensions with an uncalibrated stereo rig. In G. Sandini, editor, *Proceedings of the 2nd European Conference on Computer Vision*, volume 588 of *Lecture Notes in Computer Science*, pages 563–578, Santa Margherita Ligure, Italy, May 1992. Springer-Verlag.

[6] Olivier Faugeras. *Three-Dimensional Computer Vision: a Geometric Viewpoint.* The MIT Press, 1993.

[7] Olivier Faugeras. Stratification of 3-d vision: projective, affine, and metric representations. *Journal of the Optical Society of America A*, 12(3):465–484, March 1995.

[8] Olivier Faugeras, Tuan Luong, and Steven Maybank. Camera self-calibration: theory and experiments. In G. Sandini, editor, *Proc 2nd ECCV*, volume 588 of *Lecture Notes in Computer Science*, pages 321–334, Santa Margherita Ligure, Italy, May 1992. Springer-Verlag.

[9] Olivier Faugeras and Giorgio Toscani. Camera Calibration for 3D Computer Vision. In *International Workshop on Machine Vision and Machine Intelligence*, pages 240–247, Tokyo, February 1987.

[10] Richard Hartley. Calibration of cameras using the essential matrix. In *Proceedings of the ARPA Image Understanding Workshop*, pages 911–915. Defense Advanced Research Projects Agency, Morgan Kaufmann Publishers, Inc., 1992.

[11] Richard Hartley. Lines and points in three views-an integrated approach. In *Proceedings of the ARPA Image Understanding Workshop.* Defense Advanced Research Projects Agency, Morgan Kaufmann Publishers, Inc., 1994.

[12] Richard Hartley, Rajiv Gupta, and Tom Chang. Stereo from uncalibrated cameras. In *Proceedings of the International Conference on Computer Vision and Pattern Recognition*, pages 761–764, Urbana Champaign, IL, June 1992. IEEE.

[13] Q.-T. Luong and T. Viéville. Canonic representations for the geometries of multiple projective views. In J-O. Eklundh, editor, *Proceedings of the 3rd European Conference on Computer Vision*, volume 800-801 of *Lecture Notes in Computer Science*, pages 589–599, Vol. 1, Stockholm, Sweden, May 1994. Springer-Verlag.

[14] Quang-Tuan Luong, Rachid Deriche, Olivier Faugeras, and Théodore Papadopoulo. On determining the fundamental matrix: analysis of different methods and experimental results. In *Israelian Conf. on Artificial Intelligence and Computer Vision*, Tel-Aviv, Israel, 1993. A longer version is INRIA Tech Report RR-1894.

[15] William H. Press, Brian P. Flannery, Saul A. Teukolsky, and William T. Vetterling. *Numerical Recipes in C.* Cambridge University Press, 1988.

[16] L. Robert and O.D. Faugeras. Relative 3d positioning and 3d convex hull computation from a weakly calibrated stereo pair. In *Proceedings of the 4th*

Proc. International Conference on Computer Vision, pages 540–544, Berlin, Germany, May 1993. IEEE Computer Society Press. also INRIA Technical Report 2349.

[17] Peter Sturm and Long Quan. Affine stereo calibration. Technical Report TR 29, Lifia–Imag, 1995.

[18] T. Viéville, C. Zeller, and L. Robert. Recovering motion and structure from a set of planar patches in an uncalibrated image sequence. In *Proceedings of the International Conference on Pattern Recognition*, pages 637–641, Jerusalem, Israel, October 1994. Computer Society Press.

[19] Cyril Zeller and Olivier Faugeras. Applications of non-metric vision to some visual guided tasks. In *Proceedings of the International Conference on Pattern Recognition*, pages 132–136, Jerusalem, Israel, October 1994. Computer Society Press. A longer version in INRIA Tech Report RR2308.

[20] Cyril Zeller and Olivier Faugeras. Applications of non-metric vision to some visual guided tasks. Research Report 2308, INRIA, July 1994.

[21] Zhengyou Zhang, Rachid Deriche, Olivier Faugeras, and Quang-Tuan Luong. A robust technique for matching two uncalibrated images through the recovery of the unknown epipolar geometry. *Artificial Intelligence Journal*, 1994. to appear, also INRIA Research Report No.2273, May 1994.

Appendices

A Study of $\mathbf{S}_\lambda$

In this section, $\mathbf{R}$ is a 3×3-rotation matrix different from $\mathbf{I}_3$, θ is its angle and $\mathbf{r}$, its rotation vector such that $\mathbf{Rr} = \mathbf{r}$ and $\|\mathbf{r}\| = 1$.

Definition A.1 *We define* $\mathbf{S}_\lambda$ *by*

$$\mathbf{S}_\lambda = \mathbf{R}^2 + (\lambda - \mathrm{tr}(\mathbf{R}))\mathbf{R} + (\lambda^2 - \mathrm{tr}(\mathbf{R})\lambda + \mathrm{tr}(\mathbf{R}))\mathbf{I}_3$$

Proposition A.1

$$(\mathbf{R} - \lambda \mathbf{I}_3)\mathbf{S}_\lambda = \det(\mathbf{R} - \lambda \mathbf{I}_3)\mathbf{I}_3$$

Proof. We have

$$\det(\mathbf{R} - \lambda \mathbf{I}_3) = -\lambda^3 + \mathrm{tr}(\mathbf{R})\lambda^2 - \mathrm{tr}(\mathbf{R})\lambda + 1$$

so that, according to the Cayley-Hamilton's theorem, we have

$$\mathbf{R}^3 - \mathrm{tr}(\mathbf{R})\mathbf{R}^2 + \mathrm{tr}(\mathbf{R})\mathbf{R} - \mathbf{I}_3 = \mathbf{0}_{3\times 3}$$

Thus,

$$\begin{gathered} (\mathbf{R} - \lambda \mathbf{I}_3)\mathbf{S}_\lambda \\ = \mathbf{R}^3 - \mathrm{tr}(\mathbf{R})\mathbf{R}^2 + \mathrm{tr}(\mathbf{R})\mathbf{R} + (-\lambda^3 + \mathrm{tr}(\mathbf{R})\lambda^2 - \mathrm{tr}(\mathbf{R})\lambda)\mathbf{I}_3 \\ = \det(\mathbf{R} - \lambda \mathbf{I}_3)\mathbf{I}_3 \end{gathered}$$

□

Proposition A.2 *If* $\lambda \neq 1$, $\mathbf{S}_\lambda$ *is invertible and*

$$\mathbf{S}_\lambda^{-1} = \frac{\mathbf{R} - \lambda \mathbf{I}_3}{\det(\mathbf{R} - \lambda \mathbf{I}_3)}$$

Proof. This results from proposition A.1. □

Proposition A.3

$$\mathbf{S}_1 = (3 - \mathrm{tr}(\mathbf{R}))\mathbf{r}\mathbf{r}^T$$

Proof. Since $\mathrm{tr}(\mathbf{R}) = 2\cos(\theta) + 1$, we have[35]

$$\begin{gathered} \mathbf{S}_1 \\ = \mathbf{R}^2 - 2\cos(\theta)\mathbf{R} + \mathbf{I}_3 \\ = \mathbf{I}_3 + \sin(2\theta)[\mathbf{r}]_\times + (1 - \cos(2\theta))[\mathbf{r}]_\times^2 - 2\cos(\theta)(\mathbf{I}_3 + \sin(\theta)[\mathbf{r}]_\times + (1 - \cos(\theta))[\mathbf{r}]_\times^2) + \mathbf{I}_3 \\ = 2(1 - \cos(\theta))\mathbf{r}\mathbf{r}^T \end{gathered}$$

□

Proposition A.4 *For any* 3×3*-matrix* $\mathbf{X}$, *we have*

$$\mathbf{R}\mathbf{S}_\lambda \mathbf{X}\mathbf{S}_\lambda^T \mathbf{R}^T - \lambda^2 \mathbf{S}_\lambda \mathbf{X}\mathbf{S}_\lambda^T = \det(\mathbf{R} - \lambda \mathbf{I}_3)(\det(\mathbf{R} - \lambda \mathbf{I}_3)\mathbf{X} + \lambda \mathbf{S}_\lambda \mathbf{X} + \lambda \mathbf{X}\mathbf{S}_\lambda^T)$$

[5] using the algebraic equation $\mathbf{v}\mathbf{v}^T = \|\mathbf{v}\|^2 \mathbf{I}_3 + [\mathbf{v}]_\times^2$

Proof. Indeed, according to proposition A.1,

$$\begin{gathered}\mathbf{R}\mathbf{S}_\lambda\mathbf{X}\mathbf{S}_\lambda^T\mathbf{R}^T - \lambda^2\mathbf{S}_\lambda\mathbf{X}\mathbf{S}_\lambda^T \\ = (\det(\mathbf{R}-\lambda\mathbf{I}_3)\mathbf{I}_3 + \lambda\mathbf{S}_\lambda)\mathbf{X}(\det(\mathbf{R}-\lambda\mathbf{I}_3)\mathbf{I}_3 + \lambda\mathbf{S}_\lambda^T) - \lambda^2\mathbf{S}_\lambda\mathbf{X}\mathbf{S}_\lambda^T \\ = \det(\mathbf{R}-\lambda\mathbf{I}_3)(\det(\mathbf{R}-\lambda\mathbf{I}_3)\mathbf{X} + \lambda\mathbf{S}_\lambda\mathbf{X} + \lambda\mathbf{X}\mathbf{S}_\lambda^T)\end{gathered}$$

□

Proposition A.5 *For any λ_1 and λ_2 different from 1, we have*

$$\mathbf{S}_{\lambda_1}\mathbf{S}_{\lambda_2} = \mathbf{S}_{\lambda_2}\mathbf{S}_{\lambda_1}$$

Proof. Indeed,

$$\mathbf{S}_{\lambda_1}\mathbf{S}_{\lambda_2} = \det(\mathbf{R}-\lambda_1\mathbf{I}_3)\det(\mathbf{R}-\lambda_2\mathbf{I}_3)(\mathbf{R}^2 - (\lambda_1+\lambda_2)\mathbf{R} + \lambda_1\lambda_2\mathbf{I}_3)^{-1} = \mathbf{S}_{\lambda_2}\mathbf{S}_{\lambda_1}$$

□

Proposition A.6 *For any 3-vector $\mathbf{v}$ not proportional to $\mathbf{r}$ and any distinct λ_1 and λ_2 different from 1, $\mathbf{S}_{\lambda_1}\mathbf{v}$, $\mathbf{S}_{\lambda_2}\mathbf{v}$ and $\mathbf{r}$ are not coplanar.*

Proof. Indeed, according to proposition A.5, we have then

$$\begin{gathered}\alpha_1\mathbf{S}_{\lambda_1}\mathbf{v} + \alpha_2\mathbf{S}_{\lambda_2}\mathbf{v} + \alpha\mathbf{r} = \mathbf{0}_3 \\ \Longrightarrow \\ \alpha_1\mathbf{S}_{\lambda_2}^{-1}\mathbf{v} + \alpha_2\mathbf{S}_{\lambda_1}^{-1}\mathbf{v} + \alpha\mathbf{S}_{\lambda_1}^{-1}\mathbf{S}_{\lambda_2}^{-1}\mathbf{r} = \mathbf{0}_3 \\ \Longrightarrow \\ \left(\tfrac{\alpha_1}{\det(\mathbf{R}-\lambda_1\mathbf{I}_3)} + \tfrac{\alpha_2}{\det(\mathbf{R}-\lambda_2\mathbf{I}_3)}\right)\mathbf{R}\mathbf{v} - \left(\tfrac{\alpha_1\lambda_1}{\det(\mathbf{R}-\lambda_1\mathbf{I}_3)} + \tfrac{\alpha_2\lambda_2}{\det(\mathbf{R}-\lambda_2\mathbf{I}_3)}\right)\mathbf{v} + \tfrac{\alpha(1-\lambda_1)(1-\lambda_2)}{\det(\mathbf{R}-\lambda_1\mathbf{I}_3)\det(\mathbf{R}-\lambda_2\mathbf{I}_3)}\mathbf{r} = \mathbf{0}_3 \\ \Longrightarrow \\ \alpha = \alpha_1 = \alpha_2 = 0\end{gathered}$$

□

Proposition A.7 *For any λ different from 1, we have*

$$(\mathbf{r}\times\mathbf{S}_\lambda\mathbf{v})^T(\mathbf{r}\times\mathbf{S}_{-\lambda}\mathbf{v}) = \frac{(1-\lambda^2)^2\|\mathbf{r}\times\mathbf{v}\|^2}{\det(\mathbf{R}-\lambda\mathbf{I}_3)\det(\mathbf{R}+\lambda\mathbf{I}_3)}$$

Proof. Since we have[3]

$$\begin{aligned}\det(\mathbf{R}-\lambda\mathbf{I}_3)\det(\mathbf{R}+\lambda\mathbf{I}_3)\mathbf{S}_\lambda^{-1}\mathbf{S}_{-\lambda}^{-1T} &= (\mathbf{R}-\lambda\mathbf{I}_3)(\mathbf{R}^T+\lambda\mathbf{I}_3)\\ &= (1-\lambda^2)\mathbf{I}_3+\lambda(\mathbf{R}-\mathbf{R}^T)\\ &= (1-\lambda^2)\mathbf{I}_3+2\lambda\sin(\theta)[\mathbf{r}]_\times\end{aligned}$$

we have[1]

$$\begin{aligned}&(\mathbf{r}\times\mathbf{S}_\lambda\mathbf{v})^T(\mathbf{r}\times\mathbf{S}_{-\lambda}\mathbf{v})\\ &= (\mathbf{S}_\lambda^{-1T}[\mathbf{v}]_\times(1-\lambda)\mathbf{r})^T(\mathbf{S}_{-\lambda}^{-1T}[\mathbf{v}]_\times(1+\lambda)\mathbf{r})\\ &= (1-\lambda^2)(\mathbf{v}\times\mathbf{r})^T\mathbf{S}_\lambda^{-1}\mathbf{S}_{-\lambda}^{-1T}\mathbf{v}\times\mathbf{r}\\ &= \frac{(1-\lambda^2)^2\|\mathbf{r}\times\mathbf{v}\|^2}{\det(\mathbf{R}-\lambda\mathbf{I}_3)\det(\mathbf{R}+\lambda\mathbf{I}_3)}\end{aligned}$$

□

Proposition A.8 *For $\lambda \neq 1$, if we denote by $\mathcal{F}_\mathbf{R}$ any frame composed of $\mathbf{r}_\perp$, $\mathbf{r}\times\mathbf{r}_\perp$ and $\mathbf{r}$, where $\mathbf{r}_\perp$ is any vector orthogonal to $\mathbf{r}$, $\mathbf{S}_\lambda$ is then written in $\mathcal{F}_\mathbf{R}$:*

$$\begin{bmatrix}\sqrt{d}(1-\lambda)\mathbf{I}_2 & \mathbf{0}_2\\ \mathbf{0}_2^T & d\end{bmatrix}\begin{bmatrix}\mathbf{R}_{\theta_{\mathbf{S}_\lambda}} & \mathbf{0}_2\\ \mathbf{0}_2^T & 1\end{bmatrix}$$

where $d=\lambda^2-2\cos(\theta)\lambda+1$ and $\mathbf{R}_{\theta_{\mathbf{S}_\lambda}}$ is a the rotation matrix of angle $\theta_{\mathbf{S}_\lambda}$ defined by

$$\cos(\theta_{\mathbf{S}_\lambda})=\frac{\cos(\theta)-\lambda}{\sqrt{d}} \quad \textit{and} \quad \sin(\theta_{\mathbf{S}_\lambda})=\frac{\sin(\theta)}{\sqrt{d}}$$

Proof. First, we note that d is always positive and that

$$\det(\mathbf{R}-\lambda\mathbf{I}_3)=d(1-\lambda)$$

Then, in $\mathcal{F}_\mathbf{R}$, $\mathbf{R}$ is written

$$\begin{bmatrix}\mathbf{R}_\theta & \mathbf{0}_2\\ \mathbf{0}_2^T & 1\end{bmatrix} \quad \text{with} \quad \mathbf{R}_\theta=\begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}$$

so that $\mathbf{S}_\lambda^{-1}$ is written

$$\begin{bmatrix}\mathbf{S}_\lambda'^{-1} & \mathbf{0}_2\\ \mathbf{0}_2^T & \frac{1}{d}\end{bmatrix} \quad \text{with} \quad \mathbf{S}_\lambda'^{-1}=\frac{\mathbf{R}_\theta-\lambda\mathbf{I}_2}{d(1-\lambda)}$$

Now, we have

$$d^2(1-\lambda)^2\mathbf{S}_\lambda'^{-1}\mathbf{S}_\lambda'^{-1T}=(\mathbf{R}_\theta-\lambda\mathbf{I}_2)(\mathbf{R}_\theta^T-\lambda\mathbf{I}_2)=(\lambda^2+1)\mathbf{I}_2-\lambda(\mathbf{R}_\theta+\mathbf{R}_\theta^T)=d\mathbf{I}_2$$

This means that $\sqrt{d}(1-\lambda)\mathbf{S}'^{-1}_{\lambda}$ is a rotation matrix of angle $-\theta_{\mathbf{S}_\lambda}$ and $\mathbf{S}_{\lambda}^{-1}$ is thus written

$$\begin{bmatrix} \mathbf{R}^T_{\theta_{\mathbf{S}_\lambda}} & \mathbf{0}_2 \\ \mathbf{0}_2^T & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{d}(1-\lambda)}\mathbf{I}_2 & \mathbf{0}_2 \\ \mathbf{0}_2^T & \frac{1}{d} \end{bmatrix}$$

□

B Study of the self-calibration equations

In this section, we give the formal solutions of the equations that are at the basis of the self-calibration process described in section 5. Section B.2 deals with the equations corresponding to one displacement and section B.3 deals with the equations corresponding to two displacements, that is with systems composed of two of the preceding equations.

The unknown of these equations or systems of equations is either $\mathbf{X}$, or $(x, \mathbf{X})$, or $((x_1, x_2), \mathbf{X})$ where x, x_1 and x_2 are non-zero reals and $\mathbf{X}$ a non-zero symmetric 3×3-matrix.

These equations are defined using the following quantities:

$$\mathbf{H}_\infty = \mathbf{A}\mathbf{R}\mathbf{A}^{-1} \tag{45}$$

$$\mathbf{e}' = \mathbf{A}\mathbf{t} \tag{46}$$

$$\mathbf{F} = [\mathbf{e}']_\times \mathbf{H}_\infty \tag{47}$$

These last three equations are equivalent to the equations (5), (6) and (9) in the case where $\mathbf{A} = \mathbf{A}'$. When systems are considered, the corresponding indexed quantities are used:

$$\begin{aligned} \mathbf{H}_{\infty i} &= \mathbf{A}\mathbf{R}_i\mathbf{A}^{-1} \\ \mathbf{e}'_i &= \mathbf{A}\mathbf{t}_i \\ \mathbf{F}_i &= [\mathbf{e}'_i]_\times \mathbf{H}_{\infty i} \end{aligned}$$

We consider that $\mathbf{R}$, respectively $\mathbf{R}_i$, is equal to $\mathbf{I}_3$ and that $\mathbf{t}$, respectively $\mathbf{t}_i$, is not equal to $\mathbf{0}_3$ in which cases the solutions of the equations, respectively systems, studied are obvious.

B.1 Notations

The set of all the symmetric 3×3-matrices is a 6-dimensional vector space and $\mathbf{X}$ equally refers to the matrix or its corresponding 6-dimensional vector.

For a set of linearly independent vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_N\}$, $\mathcal{V}(\mathbf{v}_1, \ldots, \mathbf{v}_N)$ denotes the vector space of basis $(\mathbf{v}_1, \ldots, \mathbf{v}_N)$.

We denote the rotation axis of a rotation matrix $\mathbf{R}$ by any vector $\mathbf{r}$, such that $\mathbf{Rr} = \mathbf{r}$.

If $\mathbf{R}$ is a 3×3-rotation matrix and λ a real different from 1, we define $\mathbf{S}_\lambda$ by

$$\mathbf{S}_\lambda = \mathbf{R}^2 + (\lambda - \mathrm{tr}(\mathbf{R}))\mathbf{R} + (\lambda^2 - \mathrm{tr}(\mathbf{R})\lambda + \mathrm{tr}(\mathbf{R}))\mathbf{I}_3$$

$\mathbf{S}_\lambda$ is studied in section A.

B.2 The solutions corresponding to one displacement

Proposition B.1 *The solution set of equation*

$$\mathbf{RXR}^T - x^2\mathbf{X} = \mathbf{0}_{3\times 3} \tag{48}$$

is

$$\{(x, \mathbf{X}), x \in \{-1, 1\}, \mathbf{X} \in \mathcal{V}_{\mathbf{R}}\}$$

where

$$\mathcal{V}_{\mathbf{R}} = \mathcal{V}(\mathbf{I}_3, \mathbf{rr}^T)$$

Proof. Equation (48) is a homogeneous linear system in $\mathbf{X}$. Using some computer algebra system, it can be proved formerly that the rank of this system is 6 if $x^2 \neq 1$, or 4 otherwise.

If $x^2 \neq 1$, the solution is thus unique and equal to $\mathbf{0}_{3\times 3}$.

If $x^2 = 1$, the solution vector space $\mathcal{V}_{\mathbf{R}}$ is thus of dimension 2. $\mathbf{I}_3$ is a solution of equation (48) since $\mathbf{RR}^T = \mathbf{I}_3$. $\mathbf{rr}^T$ is another solution of equation (48) since $\mathbf{Rrr}^T\mathbf{R}^T = \mathbf{rr}^T$. Since, according to proposition C.5, $\mathbf{I}_3$ and $\mathbf{rr}^T$ are then linearly independent, they form a basis of $\mathcal{V}_{\mathbf{R}}$. □

Proposition B.2 *The solution set of equation*

$$\mathbf{H}_\infty \mathbf{X} \mathbf{H}_\infty^T - x^2\mathbf{X} = \mathbf{0}_{3\times 3} \tag{49}$$

is

$$\{(x, \mathbf{X}), x \in \{-1, 1\}, \mathbf{X} \in \mathcal{V}_{\mathbf{H}_\infty}\}$$

where

$$\mathcal{V}_{\mathbf{H}_\infty} = \mathcal{V}(\mathbf{AA}^T, \mathbf{Arr}^T\mathbf{A}^T)$$

Proof. According to equation (45), equation (49) is equivalent to

$$\mathbf{R}\mathbf{A}^{-1}\mathbf{X}\mathbf{A}^{-1T}\mathbf{R}^T - x^2\mathbf{A}^{-1}\mathbf{X}\mathbf{A}^{-1T} = \mathbf{0}_{3\times3}$$

Thus, $(x_0, \mathbf{X}_0)$ is a solution of equation (49) if and only if $(x_0, \mathbf{A}^{-1}\mathbf{X}_0\mathbf{A}^{-1T})$ is a solution of equation (48). ☐

Proposition B.3 *The solution set of equation*

$$[\mathbf{v}]_\times \mathbf{X} [\mathbf{v}]_\times^T = \mathbf{0}_{3\times3} \tag{50}$$

is

$$\mathcal{V}(\mathbf{v}\mathbf{v}_1^T + \mathbf{v}_1\mathbf{v}^T, \mathbf{v}\mathbf{v}_2^T + \mathbf{v}_2\mathbf{v}^T, \mathbf{v}\mathbf{v}_3^T + \mathbf{v}_3\mathbf{v}^T)$$

where $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ *is any basis of* $\mathcal{R}^3$.

Proof. Equation (50) is a homogeneous linear system in $\mathbf{X}$. Using some computer algebra system, it can be proved formerly that the rank of this system is 3. The solution vector space is thus of dimension 3.

For any vector $\mathbf{u}$ of coordinates (x_1, x_2, x_3) in $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$, on the one hand, $\mathbf{v}\mathbf{u}^T + \mathbf{u}\mathbf{v}^T$ is obviously a solution of equation (50), on the other hand, we have

$$\mathbf{v}\mathbf{u}^T + \mathbf{u}\mathbf{v}^T = \sum_{i=1}^{i=3} x_i(\mathbf{v}\mathbf{v}_i^T + \mathbf{v}_i\mathbf{v}^T)$$

Thus, $\mathbf{v}\mathbf{v}_1^T + \mathbf{v}_1\mathbf{v}^T$, $\mathbf{v}\mathbf{v}_2^T + \mathbf{v}_2\mathbf{v}^T$ and $\mathbf{v}\mathbf{v}_3^T + \mathbf{v}_3\mathbf{v}^T$ form a basis of the solution set. ☐

Proposition B.4 *The solution set of equation*

$$[\mathbf{t}]_\times(\mathbf{R}\mathbf{X}\mathbf{R}^T - x^2\mathbf{X})[\mathbf{t}]_\times^T = \mathbf{0}_{3\times3} \tag{51}$$

is

$$\{(x, \mathbf{X}), x \in \mathcal{R}, \mathbf{X} \in \mathcal{V}_{\mathbf{R},\mathbf{t},x}\}$$

where

$$\begin{aligned} \mathcal{V}_{\mathbf{R},\mathbf{t},\pm1} &= \mathcal{V}(\mathbf{I}_3, \mathbf{r}\mathbf{r}^T, \mathbf{S}_{-1}\mathbf{t}\mathbf{t}^T\mathbf{S}_{-1}^T) \\ \mathcal{V}_{\mathbf{R},\mathbf{t},x} &= \mathcal{V}(\mathbf{S}_x\mathbf{t}\mathbf{t}^T\mathbf{S}_x^T, \mathbf{S}_{-x}\mathbf{t}\mathbf{t}^T\mathbf{S}_{-x}^T, \mathbf{S}_{x^2}\mathbf{t}\mathbf{r}^T + \mathbf{r}\mathbf{t}^T\mathbf{S}_{x^2}^T) \end{aligned}$$

if $\mathbf{t}$ *is not proportional to* $\mathbf{r}$ *or*

$$\begin{aligned} \mathcal{V}_{\mathbf{R},\mathbf{t},\pm1} &= \mathcal{V}(\mathbf{I}_3, \mathbf{r}\mathbf{r}^T) \\ \mathcal{V}_{\mathbf{R},\mathbf{t},x} &= \mathcal{V}(\mathbf{r}\mathbf{r}^T) \end{aligned}$$

otherwise.

Furthermore, $\forall x \neq \pm1, \mathbf{I}_3 \notin \mathcal{V}_{\mathbf{R},\mathbf{t},x}$.

Proof. Equation (51) is a homogeneous linear system in $\mathbf{X}$. Using some computer algebra system, it can be proved formerly that the rank of this system is 3 if $\mathbf{t}$ is not proportional to $\mathbf{r}$, or otherwise, 4 if $x^2 = 1$, or 5 otherwise.

Let us first assume that $\mathbf{t}$ is proportional to $\mathbf{r}$.

If $x^2 = 1$, the solution vector space $\mathcal{V}_{\mathbf{R},\mathbf{t},\pm 1}$ is thus of dimension 2. $\mathbf{I}_3$ and $\mathbf{r}\mathbf{r}^T$, as solutions of equation (48), are also solutions of equation (51) and thus, according to proposition C.5, form a basis of $\mathcal{V}_{\mathbf{R},\mathbf{t},\pm 1}$.

If $x^2 \neq 1$, the solution vector space $\mathcal{V}_{\mathbf{R},\mathbf{t},x}$ is thus of dimension 1. $\mathbf{r}\mathbf{r}^T$ is obviously a solution of equation (51) and thus form a basis of $\mathcal{V}_{\mathbf{R},\mathbf{t},x}$.

Let us now assume that $\mathbf{t}$ is not proportional to $\mathbf{r}$. The solution vector space $\mathcal{V}_{\mathbf{R},\mathbf{t},x}$ is thus of dimension 3.

According to proposition A.4, we have

$$\begin{gathered}[\mathbf{t}]_\times(\mathbf{R}\mathbf{S}_x\mathbf{t}\mathbf{t}^T\mathbf{S}_x^T\mathbf{R}^T - x^2\mathbf{S}_x\mathbf{t}\mathbf{t}^T\mathbf{S}_x^T)[\mathbf{t}]_\times^T \\ = \det(\mathbf{R} - x\mathbf{I}_3)[\mathbf{t}]_\times(\det(\mathbf{R} - x\mathbf{I}_3)\mathbf{t}\mathbf{t}^T + x\mathbf{S}_x\mathbf{t}\mathbf{t}^T + x\mathbf{t}\mathbf{t}^T\mathbf{S}_x^T)[\mathbf{t}]_\times^T \\ = \mathbf{0}_{3\times 3}\end{gathered}$$

Let us suppose that $x^2 = 1$. In addition to $\mathbf{I}_3$ and $\mathbf{r}\mathbf{r}^T$, that is, according to proposition A.3, equivalently, $\mathbf{S}_1\mathbf{t}\mathbf{t}^T\mathbf{S}_1^T$, a third solution is thus $\mathbf{S}_{-1}\mathbf{t}\mathbf{t}^T\mathbf{S}_{-1}^T$. If $\mathbf{t}$ is not proportional to $\mathbf{r}$, $\mathbf{r}$ and $\mathbf{S}_{-1}\mathbf{t}$ are not proportional either because $\mathbf{S}_\lambda\mathbf{r}$ is proportional to $\mathbf{r}$. According to proposition C.6, $\mathbf{I}_3$, $\mathbf{r}\mathbf{r}^T$ and $\mathbf{S}_{-1}\mathbf{t}\mathbf{t}^T\mathbf{S}_{-1}^T$ are then linearly independent, thus form a basis of $\mathcal{V}_{\mathbf{R},\mathbf{t},\pm 1}$.

Let us now suppose that $x^2 \neq 1$. Then $\mathbf{S}_x\mathbf{t}\mathbf{t}^T\mathbf{S}_x^T$ and $\mathbf{S}_{-x}\mathbf{t}\mathbf{t}^T\mathbf{S}_{-x}^T$ are thus solutions. $\mathbf{S}_{x^2}\mathbf{t}\mathbf{r}^T + \mathbf{r}\mathbf{t}^T\mathbf{S}_{x^2}^T$ is then also a solution: Indeed, according to proposition A.1, we have

$$\begin{gathered}[\mathbf{t}]_\times(\mathbf{R}(\mathbf{S}_{x^2}\mathbf{t}\mathbf{r}^T + \mathbf{r}\mathbf{t}^T\mathbf{S}_{x^2}^T)\mathbf{R}^T - x^2(\mathbf{S}_{x^2}\mathbf{t}\mathbf{r}^T + \mathbf{r}\mathbf{t}^T\mathbf{S}_{x^2}^T))[\mathbf{t}]_\times^T \\ = [\mathbf{t}]_\times((\det(\mathbf{R} - x^2\mathbf{I}_3)\mathbf{I}_3 + x^2\mathbf{S}_{x^2})\mathbf{t}\mathbf{r}^T\mathbf{R}^T + \mathbf{R}\mathbf{r}\mathbf{t}^T(\det(\mathbf{R} - x^2\mathbf{I}_3)\mathbf{I}_3 + x^2\mathbf{S}_{x^2}^T) \\ -x^2(\mathbf{S}_{x^2}\mathbf{t}\mathbf{r}^T + \mathbf{r}\mathbf{t}^T\mathbf{S}_{x^2}^T))[\mathbf{t}]_\times^T \\ = \det(\mathbf{R} - x^2\mathbf{I}_3)[\mathbf{t}]_\times(\mathbf{t}\mathbf{r}^T + \mathbf{r}\mathbf{t}^T)[\mathbf{t}]_\times^T \\ = \mathbf{0}_{3\times 3}\end{gathered}$$

Since, according to proposition A.6, neither $\mathbf{S}_x\mathbf{t}$, $\mathbf{S}_{x^2}\mathbf{t}$ and $\mathbf{r}$, nor $\mathbf{S}_{-x}\mathbf{t}$, $\mathbf{S}_{x^2}\mathbf{t}$ and $\mathbf{r}$ are coplanar, $\mathbf{S}_x\mathbf{t}\mathbf{t}^T\mathbf{S}_x^T$, $\mathbf{S}_{-x}\mathbf{t}\mathbf{t}^T\mathbf{S}_{-x}^T$ and $\mathbf{S}_{x^2}\mathbf{t}\mathbf{r}^T + \mathbf{r}\mathbf{t}^T\mathbf{S}_{x^2}^T$ are linearly independent, according to proposition C.14, and thus form a basis of $\mathcal{V}_{\mathbf{R},\mathbf{t},x}$.

Now, in the case where $\mathbf{t}$ is proportional to $\mathbf{r}$, according to proposition C.5, $\forall x \neq \pm 1, \mathbf{I}_3 \notin \mathcal{V}_{\mathbf{R},\mathbf{t},x}$.

In the case where $\mathbf{t}$ is not proportional to $\mathbf{r}$ and $x \neq \pm 1$, according to proposition C.17, the rank of $(\mathbf{I}_3, \mathbf{S}_x\mathbf{t}\mathbf{t}^T\mathbf{S}_x^T, \mathbf{S}_{-x}\mathbf{t}\mathbf{t}^T\mathbf{S}_{-x}^T, \mathbf{S}_{x^2}\mathbf{t}\mathbf{r}^T + \mathbf{r}\mathbf{t}^T\mathbf{S}_{x^2}^T)$ is 4. Indeed, on the one hand, according to proposition A.6, $\mathbf{S}_x\mathbf{t}$, $\mathbf{S}_{-x}\mathbf{t}$ and $\mathbf{r}$ are then not coplanar and, on the other hand, according to proposition A.7, $(\mathbf{r}\times\mathbf{S}_x\mathbf{t})^T(\mathbf{r}\times\mathbf{S}_{-x}\mathbf{t}) \neq 0$ so that, according to proposition D.1, $\cos(\mathbf{S}_x\mathbf{t},\mathbf{S}_{-x}\mathbf{t}) \neq \cos(\mathbf{S}_x\mathbf{t},\mathbf{r})\cos(\mathbf{S}_{-x}\mathbf{t},\mathbf{r})$ and $\mathbf{S}_x\mathbf{t}$, $\mathbf{S}_{-x}\mathbf{t}$, $\mathbf{r}$ and $\mathbf{S}_{x^2}\mathbf{t}$ do not satisfy equation (70).

Consequently, $\forall x \neq \pm 1, \mathbf{I}_3 \notin \mathcal{V}_{\mathbf{R},\mathbf{t},x}$. □

Proposition B.5 *The solution set of equation*

$$\mathbf{F}\mathbf{X}\mathbf{F}^T - x^2[\mathbf{e}']_\times\mathbf{X}[\mathbf{e}']_\times^T = \mathbf{0}_{3\times 3} \tag{52}$$

is

$$\{(x,\mathbf{X}), x\in\mathcal{R}-\{0\}, \mathbf{X}\in\mathcal{V}_{\mathbf{F},x}\}$$

where

$$\begin{aligned}\mathcal{V}_{\mathbf{F},\pm 1} &= \mathcal{V}(\mathbf{A}\mathbf{A}^T, \mathbf{A}\mathbf{r}\mathbf{r}^T\mathbf{A}^T, \mathbf{A}\mathbf{S}_{-1}\mathbf{t}\mathbf{t}^T\mathbf{S}_{-1}^T\mathbf{A}^T)\\ \mathcal{V}_{\mathbf{F},x} &= \mathcal{V}(\mathbf{A}\mathbf{S}_x\mathbf{t}\mathbf{t}^T\mathbf{S}_x^T\mathbf{A}^T, \mathbf{A}\mathbf{S}_{-x}\mathbf{t}\mathbf{t}^T\mathbf{S}_{-x}^T\mathbf{A}^T, \mathbf{A}(\mathbf{S}_{x^2}\mathbf{t}\mathbf{r}^T+\mathbf{r}\mathbf{t}^T\mathbf{S}_{x^2}^T)\mathbf{A}^T)\end{aligned}$$

if $\mathbf{t}$ *is not proportional to* $\mathbf{r}$ *or*

$$\begin{aligned}\mathcal{V}_{\mathbf{F},\pm 1} &= \mathcal{V}(\mathbf{A}\mathbf{A}^T, \mathbf{A}\mathbf{r}\mathbf{r}^T\mathbf{A}^T)\\ \mathcal{V}_{\mathbf{F},x} &= \mathcal{V}(\mathbf{A}\mathbf{r}\mathbf{r}^T\mathbf{A}^T)\end{aligned}$$

otherwise.

Furthermore, $\forall x\neq\pm 1, \mathbf{A}\mathbf{A}^T \notin \mathcal{V}_{\mathbf{F},x}$.

Proof. According to equation (47), equation (52) is equivalent to

$$[\mathbf{e}']_\times(\mathbf{H}_\infty\mathbf{X}\mathbf{H}_\infty^T - x^2\mathbf{X})[\mathbf{e}']_\times^T = \mathbf{0}_{3\times 3}$$

or using equations (45) and (46)

$$[\mathbf{A}\mathbf{t}]_\times(\mathbf{A}\mathbf{R}\mathbf{A}^{-1}\mathbf{X}\mathbf{A}^{-1T}\mathbf{R}^T\mathbf{A}^T - x^2\mathbf{X})[\mathbf{A}\mathbf{t}]_\times^T = \mathbf{0}_{3\times 3}$$

that is [1]

$$[\mathbf{t}]_\times(\mathbf{R}\mathbf{A}^{-1}\mathbf{X}\mathbf{A}^{-1T}\mathbf{R}^T - x^2\mathbf{A}^{-1}\mathbf{X}\mathbf{A}^{-1T})[\mathbf{t}]_\times^T = \mathbf{0}_{3\times 3}$$

Thus, $(x_0,\mathbf{X}_0)$ is a solution of equation (52) if and only if $(x_0,\mathbf{A}^{-1}\mathbf{X}_0\mathbf{A}^{-1T})$ is a solution of equation (51). □

B.3 The solutions corresponding to two displacements

The systems considered in this section are composed of one equation of unknown $(x_1, \mathbf{X})$ and another of same type of unknown $(x_2, \mathbf{X})$, so that the unknown of the system is $(x_1, x_2, \mathbf{X})$. If $\{(x_1, \mathbf{X}), x_1 \in \mathcal{R} - \{0\}, \mathbf{X} \in \mathcal{S}_{x_1}\}$ is the solution set of the first equation and $\{(x_2, \mathbf{X}), x_2 \in \mathcal{R} - \{0\}, \mathbf{X} \in \mathcal{S}_{x_2}\}$, the solution set of the second, the solution set of the system is $\{(x_1, x_2, \mathbf{X}), x_1.x_2 \in \mathcal{R} - \{0\}, \mathbf{X} \in \mathcal{S}_{x_1} \cap \mathcal{S}_{x_2}\}$.

A basis of $\mathcal{S}_{x_1} \cap \mathcal{S}_{x_2}$ is given by $\dim(\mathcal{S}_{x_1} \cap \mathcal{S}_{x_2})$ linearly independent vectors of $\mathcal{S}_{x_1} \cap \mathcal{S}_{x_2}$ and we have[6]

$$\dim(\mathcal{S}_{x_1} \cap \mathcal{S}_{x_2}) = \dim(\mathcal{S}_{x_1}) + \dim(\mathcal{S}_{x_2}) - \dim(\mathcal{S}_{x_1} + \mathcal{S}_{x_2})$$

Proposition B.6 *The solution set of the system*

$$\begin{aligned} \mathbf{R}_1 \mathbf{X} \mathbf{R}_1^T - x_1^2 \mathbf{X} = \mathbf{0}_{3\times 3} \\ \mathbf{R}_2 \mathbf{X} \mathbf{R}_2^T - x_2^2 \mathbf{X} = \mathbf{0}_{3\times 3} \end{aligned}$$

is

$$\{(x_1, x_2, \mathbf{X}), x_1, x_2 \in \{-1, 1\}, \mathbf{X} \in \mathcal{V}_{\mathbf{R}_1} \cap \mathcal{V}_{\mathbf{R}_2}\}$$

where $\mathcal{V}_{\mathbf{R}_1} \cap \mathcal{V}_{\mathbf{R}_2}$ is $\mathcal{V}(\mathbf{I}_3)$ if $\mathbf{r}_1$ and $\mathbf{r}_2$ are not proportional, or $\mathcal{V}_{\mathbf{R}_1}$ otherwise.

Proof. According to proposition B.1, $\mathbf{I}_3$, $\mathbf{r}_1\mathbf{r}_1^T$ and $\mathbf{r}_2\mathbf{r}_2^T$ obviously span $(\mathcal{V}_{\mathbf{R}_1} + \mathcal{V}_{\mathbf{R}_2})$. Now, according to proposition C.6, if $\mathbf{r}_1$ and $\mathbf{r}_2$ are not proportional, we have

$$\dim(\mathcal{V}_{\mathbf{R}_1} \cap \mathcal{V}_{\mathbf{R}_2}) = 2 + 2 - 3 = 1$$

and $(\mathbf{I}_3)$ is obviously a basis of $\mathcal{V}_{\mathbf{R}_1} \cap \mathcal{V}_{\mathbf{R}_2}$, otherwise

$$\mathcal{V}_{\mathbf{R}_1} \cap \mathcal{V}_{\mathbf{R}_2} = \mathcal{V}_{\mathbf{R}_1} = \mathcal{V}_{\mathbf{R}_2}$$

□

Proposition B.7 *The solution set of the system*

$$\begin{aligned} \mathbf{H}_{\infty 1} \mathbf{X} \mathbf{H}_{\infty 1}^T - \mathbf{X} = \mathbf{0}_{3\times 3} \\ \mathbf{H}_{\infty 2} \mathbf{X} \mathbf{H}_{\infty 2}^T - \mathbf{X} = \mathbf{0}_{3\times 3} \end{aligned}$$

is

$$\mathcal{V}_{\mathbf{H}_{\infty 1}} \cap \mathcal{V}_{\mathbf{H}_{\infty 2}} = \begin{cases} \mathcal{V}(\mathbf{A}\mathbf{A}^T) & \text{if } \mathbf{r}_1 \text{ and } \mathbf{r}_2 \text{ are not proportional} \\ \mathcal{V}_{\mathbf{H}_{\infty 1}} & \text{otherwise} \end{cases}$$

[6] using the relation $\dim(E_1 + E_2) + \dim(E_1 \cap E_2) = \dim(E_1) + \dim(E_2)$, valid for two vector spaces E_1 and E_2

Proof. This results from proposition B.6 and by using the same argument as in the proof of proposition B.2. □

Proposition B.8 *The solution set of the system*

$$[\mathbf{t}_1]_\times(\mathbf{R}_1\mathbf{X}\mathbf{R}_1^T - x_1^2\mathbf{X})[\mathbf{t}_1]_\times^T = \mathbf{0}_{3\times 3}$$
$$[\mathbf{t}_2]_\times(\mathbf{R}_2\mathbf{X}\mathbf{R}_2^T - x_2^2\mathbf{X})[\mathbf{t}_2]_\times^T = \mathbf{0}_{3\times 3}$$

is

$$\{(x_1, x_2, \mathbf{X}), x_1, x_2 \in \Re - \{0\}, \mathbf{X} \in \mathcal{V}_{\mathbf{R}_1,\mathbf{t}_1,x_1} \cap \mathcal{V}_{\mathbf{R}_2,\mathbf{t}_2,x_2}\}$$

$\mathcal{V}_{\mathbf{R}_1,\mathbf{t}_1,\pm 1} \cap \mathcal{V}_{\mathbf{R}_2,\mathbf{t}_2,\pm 1}$ *is given in table 9 depending on the configuration of the vectors* $\mathbf{r}_1$*,* $\mathbf{S}_{1,-1}\mathbf{t}_1$*,* $\mathbf{r}_2$ *and* $\mathbf{S}_{2,-1}\mathbf{t}_2$ *between each other.*

Furthermore, we have

$$\forall x \neq \pm 1, \mathbf{I}_3 \notin \mathcal{V}_{\mathbf{R}_1,\mathbf{t}_1,x_1} \cap \mathcal{V}_{\mathbf{R}_2,\mathbf{t}_2,x_2}$$

Proof. According to proposition B.4, $\mathbf{I}_3$, $\mathbf{r}_1\mathbf{r}_1^T$, $\mathbf{S}_{1,-1}\mathbf{t}_1\mathbf{t}_1^T\mathbf{S}_{1,-1}^T$, $\mathbf{r}_2\mathbf{r}_2^T$ and $\mathbf{S}_{2,-1}\mathbf{t}_2\mathbf{t}_2^T\mathbf{S}_{2,-1}^T$ obviously span $(\mathcal{V}_{\mathbf{R}_1,\mathbf{t}_1,\pm 1} + \mathcal{V}_{\mathbf{R}_2,\mathbf{t}_2,\pm 1})$. Now, according to propositions C.5, C.6, C.8 and C.10, several cases occur that are summarized in table 9. □

Proposition B.9 *The solution set of the system*

$$\mathbf{F}_1\mathbf{X}\mathbf{F}_1^T - x_1^2[\mathbf{e}'_1]_\times\mathbf{X}[\mathbf{e}'_1]_\times^T = \mathbf{0}_{3\times 3}$$
$$\mathbf{F}_2\mathbf{X}\mathbf{F}_2^T - x_2^2[\mathbf{e}'_2]_\times\mathbf{X}[\mathbf{e}'_2]_\times^T = \mathbf{0}_{3\times 3}$$

is

$$\{(x_1, x_2, \mathbf{X}), x_1, x_2 \in \Re - \{0\}, \mathbf{X} \in \mathcal{V}_{\mathbf{F}_1,x_1} \cap \mathcal{V}_{\mathbf{F}_2,x_2}\}$$

$\mathcal{V}_{\mathbf{F}_1,\pm 1} \cap \mathcal{V}_{\mathbf{F}_2,\pm 1}$*, depending on the configuration of the vectors* $\mathbf{r}_1$*,* $\mathbf{S}_{1,-1}\mathbf{t}_1$*,* $\mathbf{r}_2$ *and* $\mathbf{S}_{2,-1}\mathbf{t}_2$ *between each other, is deduced from table 9 the following way: A basis of* $\mathcal{V}_{\mathbf{F}_1,\pm 1} \cap \mathcal{V}_{\mathbf{F}_2,\pm 1}$ *is obtained from the vectors of a basis of* $\mathcal{V}_{\mathbf{R}_1,\mathbf{t}_1,\pm 1} \cap \mathcal{V}_{\mathbf{R}_2,\mathbf{t}_2,\pm 1}$ *by left-multiplying by* $\mathbf{A}$ *and right-multiplying by* $\mathbf{A}^T$ *each of them.*

Furthermore, we have

$$\forall x \neq \pm 1, \mathbf{I}_3 \notin \mathcal{V}_{\mathbf{F}_1,x_1} \cap \mathcal{V}_{\mathbf{F}_2,x_2}$$

Proof. This results from proposition B.8 and by using the same argument as in the proof of proposition B.5. □

Configuration	$\dim(\mathcal{V}_{(\mathbf{R},\mathbf{t},\pm 1)_{1,2}})$	a basis of $\mathcal{V}_{(\mathbf{R},\mathbf{t},\pm 1)_{1,2}}$
$\mathbf{r}_1 \propto \mathbf{u}_1 \propto \mathbf{r}_2 \propto \mathbf{u}_2$	$2+2-2=2$	$\mathbf{I}_3, \mathbf{r}_1\mathbf{r}_1^T$
$\mathbf{r}_1 \propto \mathbf{u}_1 \propto \mathbf{r}_2$	$2+3-3=2$	$\mathbf{I}_3, \mathbf{r}_1\mathbf{r}_1^T$
$\mathbf{r}_1 \propto \mathbf{u}_1 \propto \mathbf{u}_2$	$2+3-3=2$	$\mathbf{I}_3, \mathbf{r}_1\mathbf{r}_1^T$
$\mathbf{r}_1 \propto \mathbf{r}_2 \propto \mathbf{u}_2$	$3+2-3=2$	$\mathbf{I}_3, \mathbf{r}_1\mathbf{r}_1^T$
$\mathbf{u}_1 \propto \mathbf{r}_2 \propto \mathbf{u}_2$	$3+2-3=2$	$\mathbf{I}_3, \mathbf{u}_1\mathbf{u}_1^T$
$\mathbf{r}_1 \propto \mathbf{u}_1, \mathbf{r}_2 \propto \mathbf{u}_2$	$2+2-3=1$	$\mathbf{I}_3$
$\mathbf{r}_1 \propto \mathbf{r}_2, \mathbf{u}_1 \propto \mathbf{u}_2$	$3+3-3=3$	$\mathbf{I}_3, \mathbf{r}_1\mathbf{r}_1^T, \mathbf{u}_1\mathbf{u}_1^T$
$\mathbf{r}_1 \propto \mathbf{u}_2, \mathbf{r}_2 \propto \mathbf{u}_1$	$3+3-3=3$	$\mathbf{I}_3, \mathbf{r}_1\mathbf{r}_1^T, \mathbf{u}_1\mathbf{u}_1^T$
$\mathbf{r}_1 \propto \mathbf{u}_1, (\mathbf{r}_1, \mathbf{r}_2, \mathbf{u}_2)$ o.b	$2+3-3=2$	$\mathbf{I}_3, \mathbf{r}_1\mathbf{r}_1^T$
$\mathbf{r}_1 \propto \mathbf{r}_2, (\mathbf{r}_1, \mathbf{u}_1, \mathbf{u}_2)$ o.b	$3+3-3=3$	$\mathbf{I}_3, \mathbf{r}_1\mathbf{r}_1^T, \mathbf{u}_1\mathbf{u}_1^T$
$\mathbf{r}_1 \propto \mathbf{u}_2, (\mathbf{r}_1, \mathbf{u}_1, \mathbf{r}_2)$ o.b	$3+3-3=3$	$\mathbf{I}_3, \mathbf{r}_1\mathbf{r}_1^T, \mathbf{u}_1\mathbf{u}_1^T$
$\mathbf{u}_1 \propto \mathbf{r}_2, (\mathbf{r}_1, \mathbf{u}_1, \mathbf{u}_2)$ o.b	$3+3-3=3$	$\mathbf{I}_3, \mathbf{r}_1\mathbf{r}_1^T, \mathbf{u}_1\mathbf{u}_1^T$
$\mathbf{u}_1 \propto \mathbf{u}_2, (\mathbf{r}_1, \mathbf{u}_1, \mathbf{r}_2)$ o.b	$3+3-3=3$	$\mathbf{I}_3, \mathbf{r}_1\mathbf{r}_1^T, \mathbf{u}_1\mathbf{u}_1^T$
$\mathbf{r}_2 \propto \mathbf{u}_2, (\mathbf{r}_1, \mathbf{u}_1, \mathbf{r}_2)$ o.b	$3+2-3=2$	$\mathbf{I}_3, \mathbf{r}_2\mathbf{r}_2^T$
$\mathbf{r}_1 \propto \mathbf{u}_1, (\mathbf{r}_1, \mathbf{r}_2, \mathbf{u}_2)$ n.o.b.	$2+3-4=1$	$\mathbf{I}_3$
$\mathbf{r}_1 \propto \mathbf{r}_2, (\mathbf{r}_1, \mathbf{u}_1, \mathbf{u}_2)$ n.o.b.	$3+3-4=2$	$\mathbf{I}_3, \mathbf{r}_1\mathbf{r}_1^T$
$\mathbf{r}_1 \propto \mathbf{u}_2, (\mathbf{r}_1, \mathbf{u}_1, \mathbf{r}_2)$ n.o.b.	$3+3-4=2$	$\mathbf{I}_3, \mathbf{r}_1\mathbf{r}_1^T$
$\mathbf{u}_1 \propto \mathbf{r}_2, (\mathbf{r}_1, \mathbf{u}_1, \mathbf{u}_2)$ n.o.b.	$3+3-4=2$	$\mathbf{I}_3, \mathbf{r}_1\mathbf{r}_1^T$
$\mathbf{u}_1 \propto \mathbf{u}_2, (\mathbf{r}_1, \mathbf{u}_1, \mathbf{r}_2)$ n.o.b.	$3+3-4=2$	$\mathbf{I}_3, \mathbf{r}_1\mathbf{r}_1^T$
$\mathbf{r}_2 \propto \mathbf{u}_2, (\mathbf{r}_1, \mathbf{u}_1, \mathbf{r}_2)$ n.o.b.	$3+2-4=1$	$\mathbf{I}_3$
$(\mathbf{r}_1, \mathbf{u}_1, \mathbf{r}_2, \mathbf{u}_2)$ harmonic	$3+3-4=2$	$\mathbf{I}_3, \frac{c(\mathbf{r}_1)}{\lVert\mathbf{r}_1\rVert^2}\mathbf{r}_1\mathbf{r}_1^T + \frac{c(\mathbf{u}_1)}{\lVert\mathbf{u}_1\rVert^2}\mathbf{u}_1\mathbf{u}_1^T$
$(\mathbf{r}_1, \mathbf{u}_1, \mathbf{r}_2, \mathbf{u}_2)$ not harmonic	$3+3-5=1$	$\mathbf{I}_3$

Table 9: Description of $\mathcal{V}_{\mathbf{R}_1,\mathbf{t}_1,\pm 1} \cap \mathcal{V}_{\mathbf{R}_2,\mathbf{t}_2,\pm 1}$ depending on the configurations of $\mathbf{r}_1$, $\mathbf{S}_{1,-1}\mathbf{t}_1$, $\mathbf{r}_2$ and $\mathbf{S}_{2,-1}\mathbf{t}_2$; we denote $\mathcal{V}_{\mathbf{R}_1,\mathbf{t}_1,\pm 1} \cap \mathcal{V}_{\mathbf{R}_2,\mathbf{t}_2,\pm 1}$ by $\mathcal{V}_{(\mathbf{R},\mathbf{t},\pm 1)_{1,2}}$, $\mathbf{S}_{1,-1}\mathbf{t}_1$ by $\mathbf{u}_1$ and $\mathbf{S}_{2,-1}\mathbf{t}_2$ by $\mathbf{u}_2$; the configurations are listed from top to bottom, from the least general case to the most general case; "$\propto$" means "is proportional to;" "o.b." means "form an orthogonal basis;" "n.o.b." means "do not form an orthogonal basis."

C Study of ranks

Proposition C.1 *The rank of* $(\mathbf{v}_1\mathbf{v}_1^T, \mathbf{v}_2\mathbf{v}_2^T)$ *is 2.*

Proof. Indeed, if $\mathbf{v}_{1\perp}$ is a vector orthogonal to $\mathbf{v}_1$ but not to $\mathbf{v}_2$, we have

$$\begin{gathered}\alpha_1\mathbf{v}_1\mathbf{v}_1^T + \alpha_2\mathbf{v}_2\mathbf{v}_2^T = \mathbf{0}_{3\times 3} \\ \Longrightarrow \\ \alpha_2(\mathbf{v}_2^T\mathbf{v}_{1\perp})\mathbf{v}_2 = \mathbf{0}_3 \\ \Longrightarrow \\ \alpha_2 = 0\end{gathered} \tag{53}$$

and equation (53) then implies that $\alpha_1 = 0$. □

Proposition C.2 *The rank of* $(\mathbf{v}_1\mathbf{v}_1^T, \mathbf{v}_2\mathbf{v}_2^T, \mathbf{v}_3\mathbf{v}_3^T)$ *is 3.*

Proof. Indeed, if $\mathbf{v}_{1\perp}$ is a vector orthogonal to $\mathbf{v}_1$ but not to $\mathbf{v}_2$ nor to $\mathbf{v}_3$, we have

$$\begin{gathered}\alpha_1\mathbf{v}_1\mathbf{v}_1^T + \alpha_2\mathbf{v}_2\mathbf{v}_2^T + \alpha_3\mathbf{v}_3\mathbf{v}_3^T = \mathbf{0}_{3\times 3} \\ \Longrightarrow \\ \alpha_2(\mathbf{v}_2^T\mathbf{v}_{1\perp})\mathbf{v}_2 + \alpha_3(\mathbf{v}_3^T\mathbf{v}_{1\perp})\mathbf{v}_3 = \mathbf{0}_3 \\ \Longrightarrow \\ \alpha_2 = \alpha_3 = 0\end{gathered} \tag{54}$$

and equation (54) then implies that $\alpha_1 = 0$. □

Proposition C.3 *If* $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ *and* $\mathbf{v}_4$ *are coplanar,*

$$\mathbf{Z} = \sum_{i=1}^{i=4} \frac{c(\mathbf{v}_i)}{\|\mathbf{v}_i\|^2}\mathbf{v}_i\mathbf{v}_i^T = \mathbf{0}_{3\times 3}$$

with, for $i, j = 1, 2, i \neq j$,

$$\begin{aligned}c(\mathbf{v}_i) &= \sin(\mathbf{v}_4,\mathbf{v}_1)\sin(\mathbf{v}_4,\mathbf{v}_2)\sin(\mathbf{v}_3,\mathbf{v}_j)^2 - \sin(\mathbf{v}_3,\mathbf{v}_1)\sin(\mathbf{v}_3,\mathbf{v}_2)\sin(\mathbf{v}_4,\mathbf{v}_j)^2 \\ c(\mathbf{v}_3) &= -\sin(\mathbf{v}_4,\mathbf{v}_1)\sin(\mathbf{v}_4,\mathbf{v}_2)\sin(\mathbf{v}_1,\mathbf{v}_2)^2 \\ c(\mathbf{v}_4) &= \sin(\mathbf{v}_3,\mathbf{v}_1)\sin(\mathbf{v}_3,\mathbf{v}_2)\sin(\mathbf{v}_1,\mathbf{v}_2)^2\end{aligned}$$

Proof. On the one hand, if $\mathbf{v}_{i\perp}$ is the vector coplanar with $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$, such that $\|\mathbf{v}_{i\perp}\| = \|\mathbf{v}_i\|$ and $(\mathbf{v}_i, \mathbf{v}_{i\perp}) = -\pi/2$, we have

$$\mathbf{v}_j^T \mathbf{v}_{i\perp} = \|\mathbf{v}_j\| \|\mathbf{v}_i\| \sin(\mathbf{v}_j, \mathbf{v}_i)$$

and thus, for $i, j = 1, 2, i \neq j$,

$$\begin{gathered}
\mathbf{Z}\mathbf{v}_{i\perp} \\
= \tfrac{\|\mathbf{v}_i\|}{\|\mathbf{v}_j\|} \sin(\mathbf{v}_j, \mathbf{v}_i) c(\mathbf{v}_j) \mathbf{v}_j + \tfrac{\|\mathbf{v}_i\|}{\|\mathbf{v}_3\|} \sin(\mathbf{v}_3, \mathbf{v}_i) c(\mathbf{v}_3) \mathbf{v}_3 + \tfrac{\|\mathbf{v}_i\|}{\|\mathbf{v}_4\|} \sin(\mathbf{v}_4, \mathbf{v}_i) c(\mathbf{v}_4) \mathbf{v}_4 \\
= \tfrac{\|\mathbf{v}_i\|}{\|\mathbf{v}_j\|} \sin(\mathbf{v}_j, \mathbf{v}_i) (\sin(\mathbf{v}_4, \mathbf{v}_1) \sin(\mathbf{v}_4, \mathbf{v}_2) \sin(\mathbf{v}_3, \mathbf{v}_i)^2 \\
- \sin(\mathbf{v}_3, \mathbf{v}_1) \sin(\mathbf{v}_3, \mathbf{v}_2) \sin(\mathbf{v}_4, \mathbf{v}_i)^2) \mathbf{v}_j \\
- \sin(\mathbf{v}_3, \mathbf{v}_i) \sin(\mathbf{v}_4, \mathbf{v}_1) \sin(\mathbf{v}_4, \mathbf{v}_2) \sin(\mathbf{v}_1, \mathbf{v}_2) \\
(\tfrac{\|\mathbf{v}_i\|}{\|\mathbf{v}_1\|} \sin(\mathbf{v}_3, \mathbf{v}_2) \mathbf{v}_1 - \tfrac{\|\mathbf{v}_i\|}{\|\mathbf{v}_2\|} \sin(\mathbf{v}_3, \mathbf{v}_1) \mathbf{v}_2) \\
+ \sin(\mathbf{v}_4, \mathbf{v}_i) \sin(\mathbf{v}_3, \mathbf{v}_1) \sin(\mathbf{v}_3, \mathbf{v}_2) \sin(\mathbf{v}_1, \mathbf{v}_2) \\
(\tfrac{\|\mathbf{v}_i\|}{\|\mathbf{v}_1\|} \sin(\mathbf{v}_4, \mathbf{v}_2) \mathbf{v}_1 - \tfrac{\|\mathbf{v}_i\|}{\|\mathbf{v}_2\|} \sin(\mathbf{v}_4, \mathbf{v}_1) \mathbf{v}_2) \\
= \mathbf{0}_3
\end{gathered}$$

On the other hand, $\mathbf{Z}(\mathbf{v}_1 \times \mathbf{v}_2) = \mathbf{0}_3$.

Consequently, since $\mathbf{v}_{1\perp}$, $\mathbf{v}_{2\perp}$ and $\mathbf{v}_1 \times \mathbf{v}_2$ form a basis of $\mathcal{R}^3$, $\mathbf{Z} = \mathbf{0}_{3\times 3}$. □

Proposition C.4 *The rank r of $(\mathbf{v}_1\mathbf{v}_1^T, \mathbf{v}_2\mathbf{v}_2^T, \mathbf{v}_3\mathbf{v}_3^T, \mathbf{v}_4\mathbf{v}_4^T)$ is:*

$$\begin{array}{rll} & 3 & \textit{if } \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \textit{ and } \mathbf{v}_4 \textit{ are coplanar,} \\ \textit{or} & 4 & \textit{otherwise.} \end{array}$$

Proof. We have

$$\alpha_1 \mathbf{v}_1 \mathbf{v}_1^T + \alpha_2 \mathbf{v}_2 \mathbf{v}_2^T + \alpha_3 \mathbf{v}_3 \mathbf{v}_3^T + \alpha_4 \mathbf{v}_4 \mathbf{v}_4^T = \mathbf{0}_{3\times 3} \tag{55}$$

$$\Longrightarrow$$

$$\alpha_3 (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \mathbf{v}_3 + \alpha_4 (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4) \mathbf{v}_4 = \mathbf{0}_3 \tag{56}$$

If $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$ are not coplanar, $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ and $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4)$ cannot be both equal to zero. So, equation (56) implies that at least $\alpha_3 = 0$ or $\alpha_4 = 0$ and, according to proposition C.2, equation (55) then implies that $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$.

If $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$ are coplanar, according to proposition C.3, r is less than 4. So, according to proposition C.2, r is 3. □

Proposition C.5 *The rank of $(\mathbf{I}_3, \mathbf{v}_1\mathbf{v}_1^T)$ is 2.*

Proof. Indeed, if $\mathbf{v}_{1\perp}$ is a vector orthogonal to $\mathbf{v}_1$, we have

$$\begin{gathered}\alpha_0 \mathbf{I}_3 + \alpha_1 \mathbf{v}_1 \mathbf{v}_1^T = \mathbf{0}_{3\times 3} \\ \Longrightarrow \\ \alpha_0 \mathbf{v}_{1\perp} = \mathbf{0}_3 \\ \Longrightarrow \\ \alpha_0 = 0\end{gathered} \tag{57}$$

and equation (57) then implies that $\alpha_1 = 0$. □

Proposition C.6 *The rank of* $(\mathbf{I}_3, \mathbf{v}_1\mathbf{v}_1^T, \mathbf{v}_2\mathbf{v}_2^T)$ *is 3.*

Proof. Indeed,

$$\begin{gathered}\alpha_0 \mathbf{I}_3 + \alpha_1 \mathbf{v}_1 \mathbf{v}_1^T + \alpha_2 \mathbf{v}_2 \mathbf{v}_2^T = \mathbf{0}_{3\times 3} \\ \Longrightarrow \\ \alpha_0 \mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{0}_3 \\ \Longrightarrow \\ \alpha_0 = 0\end{gathered} \tag{58}$$

and, according to proposition C.1, equation (58) then implies that $\alpha_1 = \alpha_2 = 0$. □

Proposition C.7 *If* $\mathbf{v}_1$*,* $\mathbf{v}_2$ *and* $\mathbf{v}_3$ *form an orthogonal basis of* $\mathcal{R}^3$*, then*

$$\mathbf{Z} = \mathbf{I}_3 - \sum_{i=1}^{i=3} \frac{1}{\|\mathbf{v}_i\|^2} \mathbf{v}_i \mathbf{v}_i^T = \mathbf{0}_{3\times 3}$$

Proof. We have

$$\forall i = 1, \ldots, 3, \mathbf{Z}\mathbf{v}_i = \mathbf{v}_i - \frac{1}{\|\mathbf{v}_i\|^2} \mathbf{v}_i \mathbf{v}_i^T \mathbf{v}_i = \mathbf{0}_3 \tag{59}$$

and since $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ form a basis of $\mathcal{R}^3$, this implies that $\mathbf{Z} = \mathbf{0}_{3\times 3}$. □

Proposition C.8 *The rank r of* $(\mathbf{I}_3, \mathbf{v}_1\mathbf{v}_1^T, \mathbf{v}_2\mathbf{v}_2^T, \mathbf{v}_3\mathbf{v}_3^T)$ *is:*

3 *if* $\mathbf{v}_1$*,* $\mathbf{v}_2$ *and* $\mathbf{v}_3$ *form an orthogonal basis of* $\mathcal{R}^3$*,*
or 4 *otherwise.*

Proof. If $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ form an orthogonal basis of $\mathcal{R}^3$, then, according to proposition C.7, r is less than 4. So, according to proposition C.6, r is 3.

Otherwise, there exists i such that $\mathbf{v}_i$ is not proportional to the crossproduct of the two others, $\mathbf{v}_j$ and $\mathbf{v}_k$. We have then

$$\begin{gathered}\alpha_0\mathbf{I}_3 + \alpha_1\mathbf{v}_1\mathbf{v}_1^T + \alpha_2\mathbf{v}_2\mathbf{v}_2^T + \alpha_3\mathbf{v}_3\mathbf{v}_3^T = \mathbf{0}_{3\times 3} \qquad (60)\\ \Longrightarrow \\ \alpha_0\mathbf{v}_j \times \mathbf{v}_k + \alpha_i(\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k)\mathbf{v}_i = \mathbf{0}_3 \\ \Longrightarrow \\ \alpha_0 = 0\end{gathered}$$

According to proposition C.2, equation (60) then implies that $\alpha_1 = \alpha_2 = \alpha_3 = 0$. □

Proposition C.9 *If* $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ *and* $\mathbf{v}_4$ *are harmonic (see section D), then*

$$\mathbf{Z} = \mathbf{I}_3 + \sum_{i=1}^{i=4} \frac{c(\mathbf{v}_i)}{\|\mathbf{v}_i\|^2}\mathbf{v}_i\mathbf{v}_i^T = \mathbf{0}_{3\times 3}$$

where, for any distinct $i, j, k, l = 1, \dots, 4$,

$$\begin{array}{lll} & c(\mathbf{v}_i) = 0 & \textit{if } \mathbf{v}_j,\ \mathbf{v}_k,\ \mathbf{v}_l \textit{ form an orthogonal basis,} \\ \textit{or} & c(\mathbf{v}_i) = -1 & \textit{if } \exists j, k = 1, \dots, 4, j \neq k, i \neq j, i \neq k, \\ & & \cos(\mathbf{v}_i, \mathbf{v}_j) = 0, \cos(\mathbf{v}_i, \mathbf{v}_k) = 0, \\ \textit{or} & c(\mathbf{v}_i) = b(\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k) & \\ & = \frac{\cos(\mathbf{v}_j, \mathbf{v}_k)}{\cos(\mathbf{v}_j, \mathbf{v}_i)\cos(\mathbf{v}_i, \mathbf{v}_k) - \cos(\mathbf{v}_j, \mathbf{v}_k)} & \textit{otherwise.} \end{array} \qquad (61)$$

Proof. Let us first suppose that

$$\forall i, j = 1, \dots, 4, i \neq j, \cos(\mathbf{v}_i, \mathbf{v}_j) \neq 0$$

Then, for any i, $c(\mathbf{v}_i)$ is defined by equation (61), which is valid since, according to proposition D.5, we have then

$$\begin{gathered}\forall i, j, k, l = 1, \dots, 4, \text{distincts}, \\ \cos(\mathbf{v}_j, \mathbf{v}_i)\cos(\mathbf{v}_i, \mathbf{v}_k) \neq \cos(\mathbf{v}_j, \mathbf{v}_k)\end{gathered}$$

$c(\mathbf{v}_i)$ really does not depend on $\mathbf{v}_j$ and $\mathbf{v}_k$ since, on the one hand, $b(\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k) = b(\mathbf{v}_i, \mathbf{v}_k, \mathbf{v}_j)$ and, on the other hand, $b(\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k) = b(\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_l)$ in virtue of equation (79). According to equation (78), for any distinct $i, j, k, l = 1, \ldots, 4$, there exists $(\lambda_k, \lambda_l) \in \mathcal{R}^2 - (0,0)$ such that

$$\mathbf{v}_i \times \mathbf{v}_j = \lambda_k \mathbf{v}_k + \lambda_l \mathbf{v}_l \tag{62}$$

and equation (62) implies that

$$\lambda_k \|\mathbf{v}_k\| \cos(\mathbf{v}_j, \mathbf{v}_k) + \lambda_l \|\mathbf{v}_l\| \cos(\mathbf{v}_j, \mathbf{v}_l) = 0 \tag{63}$$

Then, using equations (62) and (63), we have

$$\begin{array}{c}
\forall i, j, k, l = 1, \ldots, 4, \text{distincts} \\
\mathbf{Z}(\mathbf{v}_i \times \mathbf{v}_j) \\
= \mathbf{v}_i \times \mathbf{v}_j + \frac{c(\mathbf{v}_k)}{\|\mathbf{v}_k\|^2}(\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k)\mathbf{v}_k + \frac{c(\mathbf{v}_l)}{\|\mathbf{v}_l\|^2}(\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_l)\mathbf{v}_l \\
= \frac{c(\mathbf{v}_k)}{\|\mathbf{v}_k\|}(\lambda_k \|\mathbf{v}_k\|(\frac{1}{c(\mathbf{v}_k)} + 1) + \lambda_l \|\mathbf{v}_l\| \cos(\mathbf{v}_k, \mathbf{v}_l))\mathbf{v}_k \\
+ \frac{c(\mathbf{v}_l)}{\|\mathbf{v}_l\|}(\lambda_l \|\mathbf{v}_l\|(\frac{1}{c(\mathbf{v}_l)} + 1) + \lambda_k \|\mathbf{v}_k\| \cos(\mathbf{v}_l, \mathbf{v}_k))\mathbf{v}_l \\
= \frac{c(\mathbf{v}_k)\cos(\mathbf{v}_k,\mathbf{v}_l)}{\|\mathbf{v}_k\| \cos(\mathbf{v}_j,\mathbf{v}_l)}(\lambda_k \|\mathbf{v}_k\| \cos(\mathbf{v}_j, \mathbf{v}_k) + \lambda_l \|\mathbf{v}_l\| \cos(\mathbf{v}_j, \mathbf{v}_l))\mathbf{v}_k \\
+ \frac{c(\mathbf{v}_l)\cos(\mathbf{v}_l,\mathbf{v}_k)}{\|\mathbf{v}_l\| \cos(\mathbf{v}_j,\mathbf{v}_k)}(\lambda_l \|\mathbf{v}_l\| \cos(\mathbf{v}_j, \mathbf{v}_l) + \lambda_k \|\mathbf{v}_k\| \cos(\mathbf{v}_j, \mathbf{v}_k))\mathbf{v}_l \\
= \mathbf{0}_3
\end{array} \tag{64}$$

According to propositions D.3 and D.4, $\mathbf{v}_i$, $\mathbf{v}_j$ and $\mathbf{v}_k$ are not coplanar, which implies that $\mathbf{v}_i \times \mathbf{v}_j$, $\mathbf{v}_i \times \mathbf{v}_k$ and $\mathbf{v}_j \times \mathbf{v}_k$ form a basis of $\mathcal{R}^3$ and, according to equation (64), that $\mathbf{Z} = \mathbf{0}_{3\times 3}$.

Let us now suppose that

$$\exists (i,j), \cos(\mathbf{v}_i, \mathbf{v}_j) = 0$$

According to proposition D.4, it means that, among $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$, three of them, either form an orthogonal basis of $\mathcal{R}^3$, or belong to a plane whose normal is proportional to the fourth vector. If we are in the first case, there exist $i, j, k = 1, \ldots, 4$ such that $\mathbf{v}_i$, $\mathbf{v}_j$, $\mathbf{v}_k$ form an orthogonal basis, which leads to

$$c(\mathbf{v}_i) = c(\mathbf{v}_j) = c(\mathbf{v}_k) = -1 \quad , \quad c(\mathbf{v}_l) = 0$$

and the result is proven by proposition C.7. Let us suppose that we are exclusively in the second case, that is, for example, $\mathbf{v}_j$, $\mathbf{v}_k$ and $\mathbf{v}_l$ belong to a plane of normal $\mathbf{v}_i$ and no two of the three coplanar vectors are orthogonal. Then,

$$c(\mathbf{v}_i) = -1 \quad ,$$

$$c(\mathbf{v}_j) = b(\mathbf{v}_j, \mathbf{v}_k, \mathbf{v}_l) \quad , \quad c(\mathbf{v}_k) = b(\mathbf{v}_k, \mathbf{v}_j, \mathbf{v}_l) \quad , \quad c(\mathbf{v}_l) = b(\mathbf{v}_l, \mathbf{v}_j, \mathbf{v}_k)$$

Using the same developments as those occuring in equations (59) and (64), we have then

$$\mathbf{Z}\mathbf{v}_i = \mathbf{0}_3 \quad , \quad \mathbf{Z}(\mathbf{v}_i \times \mathbf{v}_j) = \mathbf{0}_3 \quad \text{and} \quad \mathbf{Z}(\mathbf{v}_i \times \mathbf{v}_k) = \mathbf{0}_3$$

Since $\mathbf{v}_i$, $\mathbf{v}_i \times \mathbf{v}_j$ and $\mathbf{v}_i \times \mathbf{v}_k$ form a basis of $\mathcal{R}^3$, this implies that $\mathbf{Z} = \mathbf{0}_{3\times 3}$. □

Proposition C.10 *The rank r of $(\mathbf{I}_3, \mathbf{v}_1\mathbf{v}_1^T, \mathbf{v}_2\mathbf{v}_2^T, \mathbf{v}_3\mathbf{v}_3^T, \mathbf{v}_4\mathbf{v}_4^T)$ is:*

4 *if $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$ are coplanar or harmonic (see section D),*
or 5 *otherwise.*

Proof. If $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$ are coplanar or harmonic, according to propositions C.3 and C.9, r is less than 5. Now, among the four vectors, $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$, there are necessarily three of them that do not form an orthogonal basis of $\mathcal{R}^3$ and so, according to proposition C.8, r is 4.

Otherwise, according to proposition D.2, there exist i and j such that $\mathbf{v}_i \times \mathbf{v}_j$ does not belong to the plane defined by the two other vectors, $\mathbf{v}_k$ and $\mathbf{v}_l$. We have then

$$\begin{gathered} \alpha_0\mathbf{I}_3 + \alpha_1\mathbf{v}_1\mathbf{v}_1^T + \alpha_2\mathbf{v}_2\mathbf{v}_2^T + \alpha_3\mathbf{v}_3\mathbf{v}_3^T + \alpha_4\mathbf{v}_4\mathbf{v}_4^T = \mathbf{0}_{3\times 3} \\ \Longrightarrow \\ \alpha_0\mathbf{v}_i \times \mathbf{v}_j + \alpha_k(\mathbf{v}_k, \mathbf{v}_i, \mathbf{v}_j)\mathbf{v}_k + \alpha_l(\mathbf{v}_l, \mathbf{v}_i, \mathbf{v}_j)\mathbf{v}_l = \mathbf{0}_3 \\ \Longrightarrow \\ \alpha_0 = 0 \end{gathered} \tag{65}$$

According to proposition C.4, equation (65) then implies that $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$. □

Proposition C.11 *The rank of $(\mathbf{v}_1\mathbf{v}_1^T, \mathbf{v}_2\mathbf{v}_3^T + \mathbf{v}_3\mathbf{v}_2^T)$ is 2.*

Proof. Indeed, if $\mathbf{v}_{2\perp}$ is a vector orthogonal to $\mathbf{v}_2$ but not to $\mathbf{v}_1$ nor to $\mathbf{v}_3$, we have

$$\begin{gathered} \alpha_1\mathbf{v}_1\mathbf{v}_1^T + \alpha_{23}(\mathbf{v}_2\mathbf{v}_3^T + \mathbf{v}_3\mathbf{v}_2^T) = \mathbf{0}_{3\times 3} \\ \Longrightarrow \\ \alpha_1(\mathbf{v}_1^T\mathbf{v}_{2\perp})\mathbf{v}_1 + \alpha_{23}(\mathbf{v}_3^T\mathbf{v}_{2\perp})\mathbf{v}_2 = \mathbf{0}_3 \\ \Longrightarrow \\ \alpha_1 = \alpha_{23} = 0 \end{gathered}$$

□

Proposition C.12 $\mathbf{v}_1\mathbf{v}_1^T$, $\mathbf{v}_2\mathbf{v}_2^T$ *and* $(\mathbf{v}_1\mathbf{v}_2^T + \mathbf{v}_2\mathbf{v}_1^T)$ *are linearly independent.*

Proof. Indeed, if $\mathbf{v}_{2\perp}$ denotes a vector orthogonal to $\mathbf{v}_2$ but not to $\mathbf{v}_1$, we have

$$\alpha_1\mathbf{v}_1\mathbf{v}_1^T + \alpha_2\mathbf{v}_2\mathbf{v}_2^T + \alpha_{12}(\mathbf{v}_1\mathbf{v}_2^T + \mathbf{v}_2\mathbf{v}_1^T) = \mathbf{0}_{3\times 3} \tag{66}$$
$$\Longrightarrow$$
$$\alpha_1(\mathbf{v}_1^T\mathbf{v}_{2\perp})\mathbf{v}_1 + \alpha_{12}(\mathbf{v}_1^T\mathbf{v}_{2\perp})\mathbf{v}_2 = \mathbf{0}_3$$
$$\Longrightarrow$$
$$\alpha_1 = \alpha_{12} = 0$$

and equation(66) then implies that $\alpha_2 = 0$. □

Proposition C.13 *If* $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ *and* $\mathbf{v}_4$ *are coplanar and such that*

$$\sin(\mathbf{v}_3,\mathbf{v}_2)\sin(\mathbf{v}_4,\mathbf{v}_1) + \sin(\mathbf{v}_4,\mathbf{v}_2)\sin(\mathbf{v}_3,\mathbf{v}_1) = 0 \tag{67}$$

then

$$\mathbf{Z} = \sum_{i=1}^{i=2}\frac{c(\mathbf{v}_i)}{\|\mathbf{v}_i\|^2}\mathbf{v}_i\mathbf{v}_i^T + \frac{c(\mathbf{v}_3,\mathbf{v}_4)}{\|\mathbf{v}_3\|\|\mathbf{v}_4\|}(\mathbf{v}_3\mathbf{v}_4^T + \mathbf{v}_4\mathbf{v}_3^T) = \mathbf{0}_{3\times 3}$$

with, for $i,j = 1,2, i \neq j$,

$$\begin{array}{rcl} c(\mathbf{v}_i) & = & 2\sin(\mathbf{v}_3,\mathbf{v}_j)\sin(\mathbf{v}_4,\mathbf{v}_j) \\ c(\mathbf{v}_3,\mathbf{v}_4) & = & -\sin(\mathbf{v}_1,\mathbf{v}_2)^2 \end{array}$$

Proof. On the one hand, if $\mathbf{v}_{i\perp}$ is the vector coplanar with $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$, such that $\|\mathbf{v}_{i\perp}\| = \|\mathbf{v}_i\|$ and $(\mathbf{v}_i,\mathbf{v}_{i\perp}) = -\pi/2$, we have

$$\mathbf{v}_j^T\mathbf{v}_{i\perp} = \|\mathbf{v}_j\|\|\mathbf{v}_i\|\sin(\mathbf{v}_j,\mathbf{v}_i)$$

and thus, for $i,j = 1,2, i \neq j$,

$$\begin{array}{c} \mathbf{Z}\mathbf{v}_{i\perp} \\ = \frac{\|\mathbf{v}_i\|}{\|\mathbf{v}_j\|}\sin(\mathbf{v}_j,\mathbf{v}_i)c(\mathbf{v}_j)\mathbf{v}_j \\ +c(\mathbf{v}_3,\mathbf{v}_4)(\frac{\|\mathbf{v}_i\|}{\|\mathbf{v}_3\|}\sin(\mathbf{v}_4,\mathbf{v}_i)\mathbf{v}_3 + \frac{\|\mathbf{v}_i\|}{\|\mathbf{v}_4\|}\sin(\mathbf{v}_3,\mathbf{v}_i)\mathbf{v}_4) \\ = 2\frac{\|\mathbf{v}_i\|}{\|\mathbf{v}_j\|}\sin(\mathbf{v}_j,\mathbf{v}_i)\sin(\mathbf{v}_3,\mathbf{v}_i)\sin(\mathbf{v}_4,\mathbf{v}_i)\mathbf{v}_j \\ -\sin(\mathbf{v}_1,\mathbf{v}_2)[\sin(\mathbf{v}_4,\mathbf{v}_i)(\frac{\|\mathbf{v}_i\|}{\|\mathbf{v}_1\|}\sin(\mathbf{v}_3,\mathbf{v}_2)\mathbf{v}_1 - \frac{\|\mathbf{v}_i\|}{\|\mathbf{v}_2\|}\sin(\mathbf{v}_3,\mathbf{v}_1)\mathbf{v}_2) \\ +\sin(\mathbf{v}_3,\mathbf{v}_i)(\frac{\|\mathbf{v}_i\|}{\|\mathbf{v}_1\|}\sin(\mathbf{v}_4,\mathbf{v}_2)\mathbf{v}_1 - \frac{\|\mathbf{v}_i\|}{\|\mathbf{v}_2\|}\sin(\mathbf{v}_4,\mathbf{v}_1)\mathbf{v}_2)] \\ = \mathbf{0}_3 \end{array}$$

On the other hand, $\mathbf{Z}(\mathbf{v}_1 \times \mathbf{v}_2) = \mathbf{0}_3$.

Consequently, since $\mathbf{v}_{1\perp}$, $\mathbf{v}_{2\perp}$ and $\mathbf{v}_1 \times \mathbf{v}_2$ form a basis of $\mathcal{R}^3$, $\mathbf{Z} = \mathbf{0}_{3\times 3}$. □

Proposition C.14 *The rank r of $(\mathbf{v}_1\mathbf{v}_1^T, \mathbf{v}_2\mathbf{v}_2^T, \mathbf{v}_3\mathbf{v}_4^T + \mathbf{v}_4\mathbf{v}_3^T)$ is:*

$$\begin{array}{rl} 2 & \text{if } \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \text{ and } \mathbf{v}_4 \text{ are coplanar and such that} \\ & \sin(\mathbf{v}_3,\mathbf{v}_2)\sin(\mathbf{v}_4,\mathbf{v}_1) + \sin(\mathbf{v}_4,\mathbf{v}_2)\sin(\mathbf{v}_3,\mathbf{v}_1) = 0 \\ \text{or } 3 & \text{otherwise.} \end{array}$$

Proof. If $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$ are coplanar and satisfy equation (67), according to proposition C.13, r is less than 3. So, according to proposition C.1, r is 2.

If $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$ are coplanar but do not satisfy equation (67), we have, for $i > 2$,

$$\mathbf{v}_i = \frac{\|\mathbf{v}_i\|\sin(\mathbf{v}_i,\mathbf{v}_2)}{\|\mathbf{v}_1\|\sin(\mathbf{v}_1,\mathbf{v}_2)}\mathbf{v}_1 - \frac{\|\mathbf{v}_i\|\sin(\mathbf{v}_i,\mathbf{v}_1)}{\|\mathbf{v}_2\|\sin(\mathbf{v}_1,\mathbf{v}_2)}\mathbf{v}_2$$

and so,

$$\begin{array}{c} \alpha_1\mathbf{v}_1\mathbf{v}_1^T + \alpha_2\mathbf{v}_2\mathbf{v}_2^T + \alpha_{34}(\mathbf{v}_3\mathbf{v}_4^T + \mathbf{v}_4\mathbf{v}_3^T) \\ = (\alpha_1 + 2\alpha_{34}\frac{\|\mathbf{v}_3\|\|\mathbf{v}_4\|\sin(\mathbf{v}_3,\mathbf{v}_2)\sin(\mathbf{v}_4,\mathbf{v}_2)}{\|\mathbf{v}_1\|^2\sin(\mathbf{v}_1,\mathbf{v}_2)^2})\mathbf{v}_1\mathbf{v}_1^T \\ +(\alpha_2 + 2\alpha_{34}\frac{\|\mathbf{v}_3\|\|\mathbf{v}_4\|\sin(\mathbf{v}_3,\mathbf{v}_1)\sin(\mathbf{v}_4,\mathbf{v}_1)}{\|\mathbf{v}_2\|^2\sin(\mathbf{v}_1,\mathbf{v}_2)^2})\mathbf{v}_2\mathbf{v}_2^T \\ -\alpha_{34}\frac{\|\mathbf{v}_3\|\|\mathbf{v}_4\|}{\|\mathbf{v}_1\|\|\mathbf{v}_2\|\sin(\mathbf{v}_1,\mathbf{v}_2)^2}(\sin(\mathbf{v}_3,\mathbf{v}_2)\sin(\mathbf{v}_4,\mathbf{v}_1) + \sin(\mathbf{v}_4,\mathbf{v}_2)\sin(\mathbf{v}_3,\mathbf{v}_1))(\mathbf{v}_1\mathbf{v}_2^T + \mathbf{v}_2\mathbf{v}_1^T) \end{array}$$

which shows, according to proposition C.12 and since $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$ do not satisfy equation (67), that $\mathbf{v}_1\mathbf{v}_1^T$, $\mathbf{v}_2\mathbf{v}_2^T$ and $\mathbf{v}_3\mathbf{v}_4^T + \mathbf{v}_4\mathbf{v}_3^T$ are linearly independent.

If $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$ are not coplanar, we have

$$\begin{array}{c} \alpha_1\mathbf{v}_1\mathbf{v}_1^T + \alpha_2\mathbf{v}_2\mathbf{v}_2^T + \alpha_{34}(\mathbf{v}_3\mathbf{v}_4^T + \mathbf{v}_4\mathbf{v}_3^T) = \mathbf{0}_{3\times3} \\ \Longrightarrow \\ \alpha_1(\mathbf{v}_1,\mathbf{v}_3,\mathbf{v}_4)\mathbf{v}_1 + \alpha_2(\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4)\mathbf{v}_2 = \mathbf{0}_3 \\ \Longrightarrow \\ \alpha_1 = 0 \quad \text{or} \quad \alpha_2 = 0 \end{array} \tag{68}$$

and equation (68) then implies that $\alpha_1 = \alpha_2 = \alpha_{23} = 0$. □

Proposition C.15 *The rank of $(\mathbf{I}_3, \mathbf{v}_1\mathbf{v}_1^T, \mathbf{v}_2\mathbf{v}_3^T + \mathbf{v}_3\mathbf{v}_2^T)$ is 3.*

Proof. Indeed, we have

$$\begin{array}{c} \alpha_0\mathbf{I}_3 + \alpha_1\mathbf{v}_1\mathbf{v}_1^T + \alpha_{23}(\mathbf{v}_2\mathbf{v}_3^T + \mathbf{v}_3\mathbf{v}_2^T) = \mathbf{0}_{3\times3} \\ \Longrightarrow \\ \alpha_0\mathbf{v}_1 \times \mathbf{v}_2 + \alpha_{23}(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)\mathbf{v}_2 = \mathbf{0}_3 \\ \Longrightarrow \\ \alpha_0 = 0 \end{array} \tag{69}$$

and, according to proposition C.11, equation (69) then implies that $\alpha_1 = \alpha_{23} = 0$. $\square$

Proposition C.16 *If* $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ *and* $\mathbf{v}_4$ *are such that*

$$\begin{gathered}\exists \varepsilon \in \{-1, 1\}, \\ \cos(\mathbf{v}_1, \mathbf{v}_3) = \varepsilon \cos(\mathbf{v}_1, \mathbf{v}_4) \quad , \quad \cos(\mathbf{v}_2, \mathbf{v}_3) = \varepsilon \cos(\mathbf{v}_2, \mathbf{v}_4) \\ \cos(\mathbf{v}_1, \mathbf{v}_2) = \cos(\mathbf{v}_1, \mathbf{v}_3)\cos(\mathbf{v}_2, \mathbf{v}_3)\end{gathered} \tag{70}$$

then

$$\mathbf{Z} = \mathbf{I}_3 + \sum_{i=1}^{i=2} \frac{c(\mathbf{v}_i)}{\|\mathbf{v}_i\|^2}\mathbf{v}_i\mathbf{v}_i^T + \frac{c(\mathbf{v}_3, \mathbf{v}_4)}{\|\mathbf{v}_3\|\|\mathbf{v}_4\|}(\mathbf{v}_3\mathbf{v}_4^T + \mathbf{v}_4\mathbf{v}_3^T) = \mathbf{0}_{3\times 3}$$

with, for $i = 1, \ldots, 2$,

$$\begin{aligned} c(\mathbf{v}_i) &= \tfrac{-1}{\sin(\mathbf{v}_i, \mathbf{v}_3)^2} \\ c(\mathbf{v}_3, \mathbf{v}_4) &= \tfrac{1}{\varepsilon - \cos(\mathbf{v}_3, \mathbf{v}_4)} \end{aligned}$$

Proof. According to proposition D.1, if $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$ satisfy equation (70), there exists $(\lambda_3, \lambda_4) \in \mathcal{R}^2 - (0, 0)$ such that

$$\mathbf{v}_1 \times \mathbf{v}_2 = \lambda_3\mathbf{v}_3 + \lambda_4\mathbf{v}_4 \tag{71}$$

and equation (71) implies that

$$\varepsilon\lambda_3\|\mathbf{v}_3\| + \lambda_4\|\mathbf{v}_4\| = 0 \tag{72}$$

Then, using equations (71) and (72), we have

$$\begin{gathered}\mathbf{Z}(\mathbf{v}_1 \times \mathbf{v}_2) \\ = \mathbf{v}_1 \times \mathbf{v}_2 + \tfrac{c(\mathbf{v}_3, \mathbf{v}_4)}{\|\mathbf{v}_3\|\|\mathbf{v}_4\|}((\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4)\mathbf{v}_3 + (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)\mathbf{v}_4) \\ = \tfrac{c(\mathbf{v}_3, \mathbf{v}_4)}{\|\mathbf{v}_3\|}(\lambda_3\|\mathbf{v}_3\|(\cos(\mathbf{v}_3, \mathbf{v}_4) + \tfrac{1}{c(\mathbf{v}_3, \mathbf{v}_4)}) + \lambda_4\|\mathbf{v}_4\|)\mathbf{v}_3 \\ + \tfrac{c(\mathbf{v}_3, \mathbf{v}_4)}{\|\mathbf{v}_4\|}(\lambda_4\|\mathbf{v}_4\|(\cos(\mathbf{v}_3, \mathbf{v}_4) + \tfrac{1}{c(\mathbf{v}_3, \mathbf{v}_4)}) + \lambda_3\|\mathbf{v}_3\|)\mathbf{v}_4 \\ = \tfrac{c(\mathbf{v}_3, \mathbf{v}_4)}{\|\mathbf{v}_3\|}(\varepsilon\lambda_3\|\mathbf{v}_3\| + \lambda_4\|\mathbf{v}_4\|)\mathbf{v}_3 \\ + \tfrac{c(\mathbf{v}_3, \mathbf{v}_4)}{\|\mathbf{v}_4\|}(\varepsilon\lambda_4\|\mathbf{v}_4\| + \lambda_3\|\mathbf{v}_3\|)\mathbf{v}_4 \\ = \mathbf{0}_3\end{gathered}$$

Similarly, there exists $(\lambda_2, \lambda_3) \in \mathcal{R}^2 - (0, 0)$ such that

$$\mathbf{v}_3 \times \mathbf{v}_1 = \lambda_2\mathbf{v}_2 + \lambda_3\mathbf{v}_3 \tag{73}$$

and equation (73) implies that

$$\varepsilon\lambda_2\|\mathbf{v}_2\|\cos(\mathbf{v}_2,\mathbf{v}_4)+\lambda_3\|\mathbf{v}_3\|=0 \tag{74}$$

Then, using equations (73) and (74), we have

$$\begin{gathered}
\mathbf{Z}(\mathbf{v}_3\times\mathbf{v}_1)\\
=\mathbf{v}_3\times\mathbf{v}_1+\tfrac{c(\mathbf{v}_2)}{\|\mathbf{v}_2\|^2}(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)\mathbf{v}_2+\tfrac{c(\mathbf{v}_3,\mathbf{v}_4)}{\|\mathbf{v}_3\|\|\mathbf{v}_4\|}(\mathbf{v}_4,\mathbf{v}_3,\mathbf{v}_1)\mathbf{v}_3\\
=\tfrac{c(\mathbf{v}_2)}{\|\mathbf{v}_2\|}(\lambda_2\|\mathbf{v}_2\|(1+\tfrac{1}{c(\mathbf{v}_2)})+\lambda_3\|\mathbf{v}_3\|\cos(\mathbf{v}_2,\mathbf{v}_3))\mathbf{v}_2\\
+\tfrac{c(\mathbf{v}_3,\mathbf{v}_4)}{\|\mathbf{v}_3\|}(\lambda_3\|\mathbf{v}_3\|(\cos(\mathbf{v}_3,\mathbf{v}_4)+\tfrac{1}{c(\mathbf{v}_3,\mathbf{v}_4)})+\lambda_2\|\mathbf{v}_2\|\cos(\mathbf{v}_2,\mathbf{v}_4))\mathbf{v}_3\\
=\tfrac{c(\mathbf{v}_2)\cos(\mathbf{v}_2,\mathbf{v}_3)}{\|\mathbf{v}_2\|}(\varepsilon\lambda_2\|\mathbf{v}_2\|\cos(\mathbf{v}_2,\mathbf{v}_4)+\lambda_3\|\mathbf{v}_3\|)\mathbf{v}_2\\
+\tfrac{c(\mathbf{v}_3,\mathbf{v}_4)}{\|\mathbf{v}_3\|}(\varepsilon\lambda_3\|\mathbf{v}_3\|+\lambda_2\|\mathbf{v}_2\|\cos(\mathbf{v}_2,\mathbf{v}_4))\mathbf{v}_3\\
=\mathbf{0}_3
\end{gathered}$$

Similarly, $\mathbf{Z}(\mathbf{v}_4\times\mathbf{v}_2)=\mathbf{0}_3$.

According to proposition D.1, neither $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$, nor $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_4$ are coplanar, which implies that $\mathbf{v}_1\times\mathbf{v}_2$, $\mathbf{v}_3\times\mathbf{v}_1$ and $\mathbf{v}_4\times\mathbf{v}_2$ form a basis of $\mathcal{R}^3$ and, thus, that $\mathbf{Z}=\mathbf{0}_{3\times3}$. □

Proposition C.17 *The rank r of $(\mathbf{I}_3,\mathbf{v}_1\mathbf{v}_1^T,\mathbf{v}_2\mathbf{v}_2^T,\mathbf{v}_3\mathbf{v}_4^T+\mathbf{v}_4\mathbf{v}_3^T)$ is:*

$$\begin{array}{rl}
3 & \textit{if } \mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3 \textit{ and } \mathbf{v}_4 \textit{ are coplanar and such that}\\
& \sin(\mathbf{v}_3,\mathbf{v}_2)\sin(\mathbf{v}_4,\mathbf{v}_1)+\sin(\mathbf{v}_4,\mathbf{v}_2)\sin(\mathbf{v}_3,\mathbf{v}_1)=0\\
& \textit{or } \mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3 \textit{ and } \mathbf{v}_4 \textit{ are such that}\\
& \exists\varepsilon\in\{-1,1\},\\
& \cos(\mathbf{v}_1,\mathbf{v}_3)=\varepsilon\cos(\mathbf{v}_1,\mathbf{v}_4)\quad,\quad\cos(\mathbf{v}_2,\mathbf{v}_3)=\varepsilon\cos(\mathbf{v}_2,\mathbf{v}_4)\\
& \cos(\mathbf{v}_1,\mathbf{v}_2)=\cos(\mathbf{v}_1,\mathbf{v}_3)\cos(\mathbf{v}_2,\mathbf{v}_3)\\
\textit{or}\quad 4 & \textit{otherwise.}
\end{array}$$

Proof. If $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$ are coplanar and satisfy equation (67) or satisfy equation (70), according to propositions C.13 and C.16, r is less than 4. So, according to proposition C.6, r is 3.

Otherwise, if $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$ are coplanar, we have

$$\begin{gathered}
\alpha_0\mathbf{I}_3+\alpha_1\mathbf{v}_1\mathbf{v}_1^T+\alpha_2\mathbf{v}_2\mathbf{v}_2^T+\alpha_{34}(\mathbf{v}_3\mathbf{v}_4^T+\mathbf{v}_4\mathbf{v}_3^T)=\mathbf{0}_{3\times3}\\
\Longrightarrow\\
\alpha_0\mathbf{v}_1\times\mathbf{v}_2=\mathbf{0}_3\\
\Longrightarrow\\
\alpha_0=0
\end{gathered}\tag{75}$$

and, according to proposition C.14, equation (75) then implies that $\alpha_1 = \alpha_2 = \alpha_{34} = 0$.

Otherwise, if $\mathbf{v}_1 \times \mathbf{v}_2$ does not belong to the plane defined by $\mathbf{v}_3$ and $\mathbf{v}_4$, equation (75) implies that

$$\begin{gathered} \alpha_0 \mathbf{v}_1 \times \mathbf{v}_2 + \alpha_{34}((\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4)\mathbf{v}_3 + (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)\mathbf{v}_4) = \mathbf{0}_3 \\ \Longrightarrow \\ \alpha_0 = \alpha_{34} = 0 \end{gathered} \tag{76}$$

According to proposition C.1, equation (75) then implies that $\alpha_1 = \alpha_2 = 0$.

Otherwise, if $\mathbf{v}_1 \times \mathbf{v}_2$ belongs to the plane defined by $\mathbf{v}_3$ and $\mathbf{v}_4$, there exists $(\lambda_3, \lambda_4) \in \mathcal{R}^2 - (0, 0)$ such that

$$\mathbf{v}_1 \times \mathbf{v}_2 = \lambda_3 \mathbf{v}_3 + \lambda_4 \mathbf{v}_4$$

and equation (76) implies that

$$(\alpha_0 \lambda_3 + \alpha_{34}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4))\mathbf{v}_3 + (\alpha_0 \lambda_4 + \alpha_{34}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3))\mathbf{v}_4 = \mathbf{0}_3 \tag{77}$$

If $\cos(\mathbf{v}_1, \mathbf{v}_3)^2 \neq \cos(\mathbf{v}_1, \mathbf{v}_4)^2$, equation (77) implies that $\alpha_0 = \alpha_{34} = 0$ and, according to proposition C.1, equation (75) then implies that $\alpha_1 = \alpha_2 = 0$. Otherwise, there exists $i = 3, \ldots, 4$ such that $\mathbf{v}_1 \times \mathbf{v}_i$ does not belong to the plane defined by $\mathbf{v}_2$ and $\mathbf{v}_i$. If we take $i = 3$, equation (75) implies that

$$\begin{gathered} \alpha_0 \mathbf{v}_1 \times \mathbf{v}_3 + \alpha_2(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)\mathbf{v}_2 + \alpha_{34}(\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4)\mathbf{v}_3 = \mathbf{0}_3 \\ \Longrightarrow \\ \alpha_0 = \alpha_{34} = 0 \quad \text{or} \quad \alpha_0 = \alpha_2 = 0 \end{gathered}$$

and, according to propositions C.1 and C.11, equation (75) then implies that $\alpha_1 = \alpha_2 = \alpha_{34} = 0$. $\square$

D Harmonic vectors

In this section, $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$ are such that

$$\forall i, \mathbf{v}_i \neq \mathbf{0}_3 \quad \text{and} \quad \forall (i,j), i \neq j, \quad \mathbf{v}_i \text{ and } \mathbf{v}_j \text{ are not proportional}$$

Proposition D.1 *For any i, j, k and l,*

$$(\mathbf{v}_i \times \mathbf{v}_j)^T(\mathbf{v}_k \times \mathbf{v}_l) = 0$$
$$\Longleftrightarrow$$
$$\cos(\mathbf{v}_i, \mathbf{v}_k)\cos(\mathbf{v}_j, \mathbf{v}_l) = \cos(\mathbf{v}_i, \mathbf{v}_l)\cos(\mathbf{v}_j, \mathbf{v}_k)$$
$$\Longleftrightarrow$$
$$(\mathbf{v}_k \times \mathbf{v}_l)^T(\mathbf{v}_i \times \mathbf{v}_j) = 0$$

Proof. Indeed, if we denote $\frac{\mathbf{v}_i}{\|\mathbf{v}_i\|}$ by $\mathbf{v}_{iN}$ and the plane defined by $\mathbf{v}_i$ and $\mathbf{v}_j$ by $P(\mathbf{v}_i, \mathbf{v}_j)$, for any distinct $i, j, k, l = 1, \ldots, 4$, we have

$$(\mathbf{v}_i \times \mathbf{v}_j)^T(\mathbf{v}_k \times \mathbf{v}_l) = 0$$
$$\Longleftrightarrow$$
$$\mathbf{v}_i \times \mathbf{v}_j \in P(\mathbf{v}_k, \mathbf{v}_l)$$
$$\Longleftrightarrow$$
$$\mathbf{v}_{iN} \times \mathbf{v}_{jN} \in P(\mathbf{v}_{kN}, \mathbf{v}_{lN})$$
$$\Longleftrightarrow$$
$$\exists(\lambda_k, \lambda_l) \in \mathcal{R}^2 - (0,0), \mathbf{v}_{iN} \times \mathbf{v}_{jN} = \lambda_k \mathbf{v}_{kN} + \lambda_l \mathbf{v}_{lN}$$
$$\Longleftrightarrow$$
$$\exists(\lambda_k, \lambda_l) \in \mathcal{R}^2 - (0,0),$$
$$\mathbf{v}_{iN}^T(\lambda_k \mathbf{v}_{kN} + \lambda_l \mathbf{v}_{lN}) = 0 \quad \text{and} \quad \mathbf{v}_{jN}^T(\lambda_k \mathbf{v}_{kN} + \lambda_l \mathbf{v}_{lN}) = 0$$
$$\Longleftrightarrow$$
$$(\mathbf{v}_{iN}^T \mathbf{v}_{kN})(\mathbf{v}_{jN}^T \mathbf{v}_{lN}) = (\mathbf{v}_{iN}^T \mathbf{v}_{lN})(\mathbf{v}_{jN}^T \mathbf{v}_{kN})$$
$$\Longleftrightarrow$$
$$\cos(\mathbf{v}_i, \mathbf{v}_k)\cos(\mathbf{v}_j, \mathbf{v}_l) = \cos(\mathbf{v}_i, \mathbf{v}_l)\cos(\mathbf{v}_j, \mathbf{v}_k)$$

□

Proposition D.2 $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ *and* $\mathbf{v}_4$ *are said to be* harmonic *if and only if they satisfy one of the three following equivalent equations:*

$$\forall i, j, k, l = 1, \ldots, 4, \textit{distincts}, (\mathbf{v}_i \times \mathbf{v}_j)^T(\mathbf{v}_k \times \mathbf{v}_l) = 0 \tag{78}$$

$$\begin{gathered} \forall i, j, k, l = 1, \ldots, 4, \textit{distincts}, \\ \cos(\mathbf{v}_i, \mathbf{v}_k)\cos(\mathbf{v}_j, \mathbf{v}_l) = \cos(\mathbf{v}_i, \mathbf{v}_l)\cos(\mathbf{v}_j, \mathbf{v}_k) \end{gathered} \tag{79}$$

$$\begin{aligned} \cos(\mathbf{v}_1, \mathbf{v}_2)\cos(\mathbf{v}_3, \mathbf{v}_4) &= \cos(\mathbf{v}_1, \mathbf{v}_3)\cos(\mathbf{v}_2, \mathbf{v}_4) \\ &= \cos(\mathbf{v}_1, \mathbf{v}_4)\cos(\mathbf{v}_2, \mathbf{v}_3) \end{aligned} \tag{80}$$

Proof. Equation (79) is clearly equivalent to equation (80) and, according to proposition D.1, equation (78) is equivalent to equation (79). □

Proposition D.3 *If $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ belong to a plane P then*

$$\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3 \textit{ and } \mathbf{v}_4 \textit{ are harmonic} \tag{81}$$

$$\Longleftrightarrow$$

$$\mathbf{v}_4 \textit{ is proportional to the normal to } P \tag{82}$$

Proof. If $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ belong to P, the normal to P, $\mathbf{v}_1 \times \mathbf{v}_2$ and $\mathbf{v}_1 \times \mathbf{v}_3$ are all proportional. If $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$ are harmonic, equation (78) then shows that this normal is proportional to the intersection of the plane defined by $\mathbf{v}_3$ and $\mathbf{v}_4$ and the plane defined by $\mathbf{v}_2$ and $\mathbf{v}_4$, that is $\mathbf{v}_4$.

Inversely, if $\mathbf{v}_4$ is proportional to the normal to P then $\cos(\mathbf{v}_1, \mathbf{v}_4) = \cos(\mathbf{v}_2, \mathbf{v}_4) = \cos(\mathbf{v}_3, \mathbf{v}_4) = 0$ and $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$ are harmonic according to equation (80). □

Proposition D.4

$$\begin{gathered}\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3 \textit{ and } \mathbf{v}_4 \textit{ are harmonic and}\\ \exists (i,j), \cos(\mathbf{v}_i, \mathbf{v}_j) = 0\end{gathered} \tag{83}$$

$$\Longleftrightarrow$$

$$\begin{gathered}\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3 \textit{ and } \mathbf{v}_4 \textit{ are harmonic and}\\ \exists (i,j,k), \textit{distincts}, \cos(\mathbf{v}_j, \mathbf{v}_i)\cos(\mathbf{v}_i, \mathbf{v}_k) = \cos(\mathbf{v}_j, \mathbf{v}_k) = 0\end{gathered} \tag{84}$$

$$\Longleftrightarrow$$

$$\begin{gathered}\textit{Among the four vectors, } \mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3 \textit{ and } \mathbf{v}_4, \textit{ three of them,}\\ \textit{either form an orthogonal basis of } \mathcal{R}^3,\\ \textit{or belong to a plane whose normal is proportional to the fourth vector.}\end{gathered} \tag{85}$$

Proof. If assertion (84) is true then assertion (83) is obviously true also.

Inversely, if assertion (83) is true then equation (79) shows that

$$\cos(\mathbf{v}_i, \mathbf{v}_j) = 0 \Longrightarrow \cos(\mathbf{v}_k, \mathbf{v}_j) = 0 \quad \text{or} \quad \cos(\mathbf{v}_i, \mathbf{v}_l) = 0$$

so that

$$\begin{aligned} &\cos(\mathbf{v}_i, \mathbf{v}_j) = 0 \Longrightarrow \\ &\cos(\mathbf{v}_i, \mathbf{v}_k)\cos(\mathbf{v}_k, \mathbf{v}_j) = \cos(\mathbf{v}_i, \mathbf{v}_j) = 0 \\ \text{or} \quad &\cos(\mathbf{v}_i, \mathbf{v}_l)\cos(\mathbf{v}_l, \mathbf{v}_j) = \cos(\mathbf{v}_i, \mathbf{v}_j) = 0 \end{aligned}$$

and assertion (84) is true.

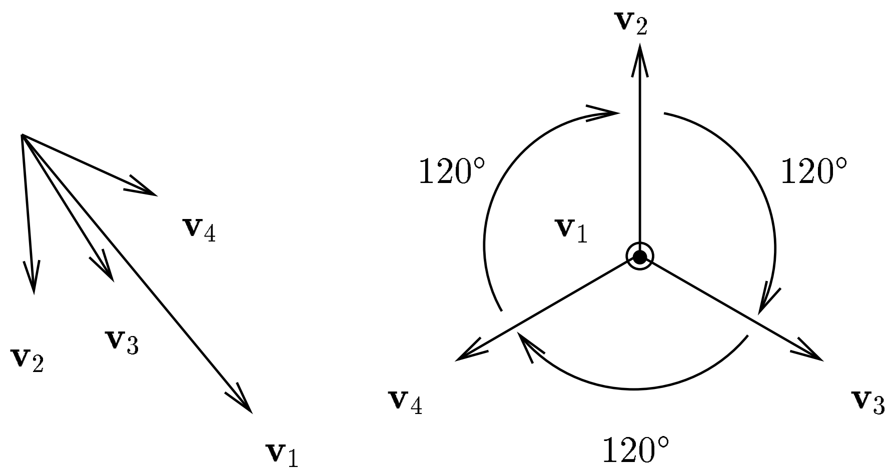


Figure 15: The umbrella from a general viewpoint on left and from above on right.

Now, if assertion (83) is true then at least one cosinus in each member of equation (80) is equal to zero and each of the sixteen possibilities leads to one of the two configurations described in assertion (85).

Inversely, if assertion (85) is true then at least one cosinus in each member of equation (80) is equal to zero and thus assertion (83) is also true. □

Another example of four harmonic vectors is given by four vectors that form an "umbrella", that is such that three of them have same angle α between each other and belong to a cone of angle β whose axis is the fourth vector (see figure 15): indeed, each member of equation (80) is then equal to $\cos(\alpha)\cos(\beta)$.

Proposition D.5 *If* $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ *and* $\mathbf{v}_4$ *are harmonic,*

$$\cos(\mathbf{v}_j, \mathbf{v}_i)\cos(\mathbf{v}_i, \mathbf{v}_k) = \cos(\mathbf{v}_j, \mathbf{v}_k) \tag{86}$$

$$\Longleftrightarrow$$

$$\cos(\mathbf{v}_j, \mathbf{v}_i)\cos(\mathbf{v}_i, \mathbf{v}_k) = \cos(\mathbf{v}_j, \mathbf{v}_k) = 0 \tag{87}$$

$$\tag{88}$$

Proof. According to equation (78), there exists $(\lambda_i, \lambda_k) \in \mathcal{R}^2 - (0,0)$ such that

$$\mathbf{v}_j \times \mathbf{v}_l = \lambda_i \mathbf{v}_i + \lambda_k \mathbf{v}_k \tag{89}$$

which implies that

$$\lambda_i \|\mathbf{v}_i\| \cos(\mathbf{v}_j, \mathbf{v}_i) + \lambda_k \|\mathbf{v}_k\| \cos(\mathbf{v}_j, \mathbf{v}_k) = 0 \tag{90}$$

Using equations (89) and (90), we have

$$\begin{aligned} & \cos(\mathbf{v}_j, \mathbf{v}_i)\mathbf{v}_i^T(\mathbf{v}_j \times \mathbf{v}_l) \\ = {} & \lambda_i \|\mathbf{v}_i\|^2 \cos(\mathbf{v}_j, \mathbf{v}_i) + \lambda_k \|\mathbf{v}_i\| \|\mathbf{v}_k\| \cos(\mathbf{v}_j, \mathbf{v}_i) \cos(\mathbf{v}_i, \mathbf{v}_k) \\ = {} & \lambda_k \|\mathbf{v}_i\| \|\mathbf{v}_k\| (\cos(\mathbf{v}_j, \mathbf{v}_i) \cos(\mathbf{v}_i, \mathbf{v}_k) - \cos(\mathbf{v}_j, \mathbf{v}_k)) \end{aligned}$$

So equation (86) implies that $\cos(\mathbf{v}_j, \mathbf{v}_i) = 0$ or $\mathbf{v}_i^T(\mathbf{v}_j \times \mathbf{v}_l) = 0$. If $\cos(\mathbf{v}_j, \mathbf{v}_i) = 0$, equation (87) is satisfied. $\mathbf{v}_i^T(\mathbf{v}_j \times \mathbf{v}_l) = 0$ means that $\mathbf{v}_i$, $\mathbf{v}_j$ and $\mathbf{v}_l$ belong to a plane P and, according to proposition D.3, the normal to P is necessarily $\mathbf{v}_k$. Thus, $\cos(\mathbf{v}_i, \mathbf{v}_k) = 0$ and assertion (87) is true again. $\square$

Unité de recherche INRIA Lorraine, Technopôle de Nancy-Brabois, Campus scientifique,
615 rue du Jardin Botanique, BP 101, 54600 VILLERS LÈS NANCY
Unité de recherche INRIA Rennes, Irisa, Campus universitaire de Beaulieu, 35042 RENNES Cedex
Unité de recherche INRIA Rhône-Alpes, 46 avenue Félix Viallet, 38031 GRENOBLE Cedex 1
Unité de recherche INRIA Rocquencourt, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex
Unité de recherche INRIA Sophia-Antipolis, 2004 route des Lucioles, BP 93, 06902 SOPHIA-ANTIPOLIS Cedex

Éditeur
INRIA, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)
ISSN 0249-6399